Chapter 6: Random Variables

Mr Faruk
Teacher of Mathematics
BSc/MSc/PGCE Mathematics
cieigcsesolutions@gmail.com

S1 Chapter 6 Solutions

3. The discrete random variable X has probability distribution

$$P(X = x) = \frac{1}{5} \qquad x = 1, 2, 3, 4, 5$$

(a) Write down the name given to this distribution.

(1)

Find

(b) $P(X = 4)$

(1)

(c) $F(3)$

(1)

(d) $P(3X - 3 > X + 4)$

(2)

(e) Write down $E(X)$

(1)

(f) Find $E(X^2)$

(2)

(g) Hence find $Var(X)$

(2)

Given that $E(aX - 3) = 11.4$

(h) find $Var(aX - 3)$

(4)

5. A biased tetrahedral die has faces numbered 0, 1, 2 and 3. The die is rolled and the number face down on the die, X, is recorded. The probability distribution of X is

x	0	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

If $X = 3$ then the final score is 3
If $X \neq 3$ then the die is rolled again and the final score is the sum of the two numbers.

The random variable T is the final score.

(a) Find $P(T = 2)$
(2)

(b) Find $P(T = 3)$
(3)

(c) Given that the die is rolled twice, find the probability that the final score is 3
(3)

2. The discrete random variable X has the following probability distribution.

x	-2	-1	0	1	3
$P(X = x)$	0.15	a	b	a	0.4

(a) Find $E(X)$.

(2)

Given that $E(X^2) = 4.54$

(b) find the value of a and the value of b.

(5)

The random variable $Y = 3 - 2X$

(c) Find $\text{Var}(Y)$.

(3)

5. Some children are playing a game involving throwing a ball into a bucket. Each child has 3 throws and the number of times the ball lands in the bucket, x, is recorded. Their results are given in the table below.

x	0	1	2	3
Frequency	16	36	24	4

(a) Find $\bar{x}$

(1)

Sandra decides to model the game by assuming that on each throw, the probability of the ball landing in the bucket is 0.4 for every child on every throw and that the throws are all independent. The random variable S represents the number of times the ball lands in the bucket for a randomly selected child.

(b) Find $P(S = 2)$

(2)

(c) Complete the table below to show the probability distribution for S.

s	0	1	2	3
$P(S = s)$		0.432		0.064

(1)

Ting believes that the probability of the ball landing in the bucket is not the same for each throw. He suggests that the probability will increase with each throw and uses the model

$$p_i = 0.15i + 0.10$$

where $i = 1, 2, 3$ and p_i is the probability that the ith throw of the ball, by any particular child, will land in the bucket.
The random variable T represents the number of times the ball lands in the bucket for a randomly selected child using Ting's model.

(d) Show that

(i) $P(T = 3) = 0.055$

(ii) $P(T = 1) = 0.45$

(5)

(e) Complete the table below to show the probability distribution for T, stating the exact probabilities in each case.

t	0	1	2	3
$P(T = t)$		0.45		0.055

(3)

(f) State, giving your reasons, whether Sandra's model or Ting's model is the more appropriate for modelling this game.

(3)

5. The discrete random variable X represents the score when a biased spinner is spun. The probability distribution of X is given by

x	-2	-1	0	2	3
$P(X = x)$	p	p	q	$\frac{1}{4}$	p

where p and q are probabilities.

(a) Find $E(X)$.

(2)

Given that $\text{Var}(X) = 2.5$

(b) find the value of p.

(5)

(c) Hence find the value of q.

(1)

Amar is invited to play a game with the spinner.
The spinner is spun once and X_1 is the score on the spinner.

If $X_1 > 0$ Amar wins the game.

If $X_1 = 0$ Amar loses the game.

If $X_1 < 0$ the spinner is spun again and X_2 is the score on this second spin and if $X_1 + X_2 > 0$ Amar wins the game, otherwise Amar loses the game.

(d) Find the probability that Amar wins the game.

(4)

Amar does not want to lose the game.
He says that because $E(X) > 0$ he will play the game.

(e) State, giving a reason, whether or not you would agree with Amar.

(2)

7. The number of cakes, X, bought by customers at a particular shop has probability distribution

x	1	2	3	4	5	6	>6
$P(X = x)$	0.35	a	a	0.15	b	b	0

where a and b are constants.

Given that $E(X) = 2.5$

(a) (i) show that $a = 0.2$

(ii) find the value of b

(5)

(b) Calculate $\text{Var}(4X + 3)$

(4)

The cost to produce each cake is 20 cents and the shopkeeper sells each cake for 80 cents.

(c) Find the expected profit made by the shopkeeper for a randomly selected customer buying cakes.

(2)

The shopkeeper decides to run a promotion where she gives 1 free cake to all customers who buy 4 or more cakes. During the promotion the number of cakes, Y, taken away by a customer buying cakes has the following probability distribution

y	1	2	3	4	5	6
$P(Y = y)$	$\frac{3}{40}$	$\frac{4}{40}$	$\frac{3}{40}$	0	$\frac{22}{40}$	$\frac{8}{40}$

(d) Find the expected profit made by the shopkeeper for a randomly selected customer buying cakes during the promotion.

(4)

1. The discrete random variable X has the following probability distribution

x	-2	-1	1	3	4
$P(X = x)$	0.15	a	b	c	0.15

where a, b and c are probabilities.

The mean value of X is 1 and $F(1) = 0.63$

Find the value of a, the value of b and the value of c.

(5)

6. A tennis tournament has 5 rounds. After each round, winners go into the next round and losers are knocked out of the tournament. To enter the tournament players must pay an entry fee of \$10 but only the person who wins all 5 rounds receives the prize of \$260

Serena enters this tennis tournament. The random variable S represents the total number of rounds Serena wins. The probability distribution for S is given in the following table.

s	0	1	2	3	4	5
$P(S = s)$	k	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{5}$	$\frac{k}{6}$

(a) Show that $k = \frac{20}{49}$

(2)

(b) Find $E(S)$

(3)

(c) Find Serena's expected profit if she enters the tennis tournament.

(3)

Roger also enters this tennis tournament. Given that Roger is still in the tournament, the probability that he wins the next round is a constant p.
The random variable R represents the total number of rounds that Roger wins.

(d) Explain why $P(R = 2) = p^2(1 - p)$

(2)

(e) Find, in terms of p, the probability distribution for R.

(3)

(f) Find the smallest value of p such that Roger's expected profit is at least as great as Serena's.

(4)

1. The discrete random variable X takes the values -1, 2, 3, 4 and 7 only.

Given that

$$P(X = x) = \frac{8 - x}{k} \quad \text{for } x = -1, 2, 3, 4 \text{ and } 7$$

find the value of $E(X)$

(5)

6. The random variable A represents the score when a spinner is spun. The probability distribution for A is given in the following table.

a	1	4	5	7
$P(A = a)$	0.40	0.20	0.25	0.15

(a) Show that $E(A) = 3.5$

(2)

(b) Find $\text{Var}(A)$

(3)

The random variable B represents the score on a 4-sided die. The probability distribution for B is given in the following table where k is a positive integer.

b	1	3	4	k
$P(B = b)$	0.25	0.25	0.25	0.25

(c) Write down the name of the probability distribution of B.

(1)

(d) Given that $E(B) = E(A)$ state, giving a reason, the value of k.

(1)

The random variable $X \sim N(\mu, \sigma^2)$

Sam and Tim are playing a game with the spinner and the die.

They each spin the spinner once to obtain their value of A and each roll the die once to obtain their value of B.
Their value of A is taken as their value of μ and their value of B is taken as their value of σ.
The person with the larger value of $P(X > 3.5)$ is the winner.

(e) Given that Sam obtained values of $a = 4$ and $b = 3$ and Tim obtained $b = 4$ find, giving a reason, the probability that Tim wins.

(2)

(f) Find the largest value of $P(X > 3.5)$ achievable in this game.

(4)

(g) Find the probability of achieving this value.

(2)

4. A spinner can land on the numbers 10, 12, 14 and 16 only and the probability of the spinner landing on each number is the same.
The random variable X represents the number that the spinner lands on when it is spun once.

(a) State the name of the probability distribution of X.

(1)

(b) (i) Write down the value of $E(X)$

(1)

(ii) Find $Var(X)$

(2)

A second spinner can land on the numbers 1, 2, 3, 4 and 5 only.
The random variable Y represents the number that this spinner lands on when it is spun once. The probability distribution of Y is given in the table below

y	1	2	3	4	5
$P(Y = y)$	$\frac{4}{30}$	$\frac{9}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$

(c) Find (i) $E(Y)$

(2)

(ii) $Var(Y)$

(3)

The random variable $W = aX + b$, where a and b are constants and $a > 0$
Given that $E(W) = E(Y)$ and $Var(W) = Var(Y)$

(d) find the value of a and the value of b.

(5)

Each of the two spinners is spun once.

(e) Find $P(W = Y)$

(2)

5. The discrete random variable X has the following probability distribution

x	-2	-1	0	1	4
$P(X = x)$	a	b	c	b	a

Given that $E(X) = 0.5$

(a) find the value of a.

(2)

Given also that $\text{Var}(X) = 5.01$

(b) find the value of b and the value of c.

(5)

The random variable $Y = 5 - 8X$

(c) Find (i) $E(Y)$

(ii) $\text{Var}(Y)$

(3)

(d) Find $P(4X^2 > Y)$

(5)

5. The discrete random variable Y has the following probability distribution

y	-9	-5	0	5	9
$P(Y = y)$	q	r	u	r	q

where q, r and u are probabilities.

(a) Write down the value of $E(Y)$

(1)

The cumulative distribution function of Y is $F(y)$

Given that $F(0) = \frac{19}{30}$

(b) show that the value of u is $\frac{4}{15}$

(3)

Given also that $\text{Var}(Y) = 37$

(c) find the value of q and the value of r

(4)

The coordinates of a point P are $(12, Y)$

The random variable D represents the length of OP

(d) Find the probability distribution of D

(6)

4. The random variable W has a discrete uniform distribution where

$$P(W = w) = \frac{1}{5} \qquad \text{for } w = 1, 2, 3, 4, 5$$

(a) Find $P(2 \leqslant W < 3.5)$

(1)

The discrete random variable $X = 5 - 2W$

(b) Find $E(X)$

(3)

(c) Find $P(X < W)$

(2)

The discrete random variable $Y = \dfrac{1}{W}$

(d) Find

(i) the probability distribution of Y

(ii) $\text{Var}(Y)$, showing your working.

(5)

(e) Find $\text{Var}(2 - 3Y)$

(2)

7. A bag contains n marbles of which 7 are green.

From the bag, 3 marbles are selected at random.

The random variable X represents the number of green marbles selected.

The cumulative distribution function of X is given by

x	0	1	2	3
$F(x)$	a	b	$\frac{37}{38}$	1

(a) Show that $n(n-1)(n-2) = 7980$
(4)

(b) Verify that $n = 21$ satisfies the equation in part (a).
(1)

Given that $n = 21$

(c) find the exact value of a and the exact value of b
(6)

5. A red spinner is designed so that the score R is given by the following probability distribution.

r	2	3	4	5	6
$P(R = r)$	0.25	0.3	0.15	0.1	0.2

(a) Show that $E(R^2) = 15.8$

(1)

Given also that $E(R) = 3.7$

(b) find the standard deviation of R, giving your answer to 2 decimal places.

(2)

A yellow spinner is designed so that the score Y is given by the probability distribution in the table below. The cumulative distribution function $F(y)$ is also given.

y	2	3	4	5	6
$P(Y = y)$	0.1	0.2	0.1	a	b
$F(y)$	0.1	0.3	0.4	c	d

(c) Write down the value of d

(1)

Given that $E(Y) = 4.55$

(d) find the value of c

(5)

Pabel and Jessie play a game with these two spinners.
Pabel uses the red spinner.
Jessie uses the yellow spinner.
They take turns to spin their spinner.
The winner is the first person whose spinner lands on the number 2 and the game ends.
Jessie spins her spinner first.

(e) Find the probability that Jessie wins on her second spin.

(2)

(f) Calculate the probability that, in a game, the score on Pabel's first spin is the same as the score on Jessie's first spin.

(3)

4. The cumulative distribution function of the discrete random variable W, which takes only the values 6, 7 and 8, is given by

$$F(W) = \frac{(w+3)(w-1)}{77} \quad \text{for } w = 6, 7, 8$$

Find $E(W)$

(4)

7. Adana selects one number at random from the distribution of X which has the following probability distribution.

x	0	5	10
$P(X = x)$	0.1	0.2	0.7

(a) Given that the number selected by Adana is not 5, write down the probability it is 0 **(1)**

(b) Show that $E(X^2) = 75$ **(1)**

(c) Find $\text{Var}(X)$ **(3)**

(d) Find $\text{Var}(4 - 3X)$ **(2)**

Bruno and Charlie each independently select one number at random from the distribution of X

(e) Find the probability that the number Bruno selects is greater than the number Charlie selects. **(3)**

Devika multiplies Bruno's number by Charlie's number to obtain a product, D

(f) Determine the probability distribution of D **(4)**

3. The probability distribution of the discrete random variable X is given by

x	2	3	4
$P(X = x)$	a	0.4	$0.6 - a$

where a is a constant.

(a) Find, in terms of a, $E(X)$

(2)

(b) Find the range of the possible values of $E(X)$

(3)

Given that $Var(X) = 0.56$

(c) find the possible values of a

(6)

5. A discrete random variable Y has probability function

$$P(Y = y) = \begin{cases} k(3 - y) & y = 1, 2 \\ k(y^2 - 8) & y = 3, 4, 5 \\ k & y = 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{30}$ **(2)**

Find the exact value of

(b) $P(1 < Y \leqslant 4)$ **(2)**

(c) $E(Y)$ **(2)**

The random variable $X = 15 - 2Y$

(d) Calculate $P(Y \geqslant X)$ **(3)**

(e) Calculate $\text{Var}(X)$ **(4)**

4. The discrete random variable X has the following probability distribution.

x	1	2	3	4
$P(X=x)$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

(a) Show that $E\left(\frac{1}{X}\right) = \frac{2}{5}$ **(1)**

(b) Find $\text{Var}\left(\frac{1}{X}\right)$ **(3)**

The random variable $Y = \frac{30}{X}$

(c) Find

(i) $E(Y)$

(ii) $\text{Var}(Y)$ **(3)**

(d) Find $P(X < 3 \mid Y < 20)$ **(5)**

7. The cumulative distribution of a discrete random variable X is given by

x	1	2	3	4
$\mathrm{F}(x)$	$\frac{1}{13}$	$\frac{2k-1}{26}$	$\frac{3(k+1)}{26}$	$\frac{k+4}{8}$

where k is a positive constant.

(a) Show that $k = 4$

(1)

(b) Find the probability distribution of the discrete random variable X

(3)

(c) Using your answer to part (b), write down the mode of X

(1)

(d) Calculate $\mathrm{Var}(13X - 6)$

(5)

2. A spinner can land on the numbers 2, 4, 5, 7 or 8 only.
The random variable X represents the number that this spinner lands on when it is spun once. The probability distribution of X is given in the table below.

x	2	4	5	7	8
$\mathrm{P}(X = x)$	0.25	0.3	0.2	0.1	0.15

(a) Find $\mathrm{P}(2X - 3 > 5)$

(1)

Given that $\mathrm{E}(X) = 4.6$

(b) show that $\mathrm{Var}(X) = 4.14$

(3)

The random variable $Y = aX - b$ where a and b are positive constants.

Given that

$$\mathrm{E}(Y) = 13.4 \quad \text{and} \quad \mathrm{Var}(Y) = 66.24$$

(c) find the value of a and the value of b

(4)

In a game Sam and Alex each spin the spinner once, landing on X_1 and X_2 respectively.

Sam's score is given by the random variable $S = X_1$
Alex's score is given by the random variable $R = 2X_2 - 3$

The person with the higher score wins the game. If the scores are the same it is a draw.

(d) Find the probability that Sam wins the game.

(4)

6. A biased die with six faces is rolled. The discrete random variable X represents the score which is uppermost. The **cumulative** distribution function of X is shown in the table below.

x	1	2	3	4	5	6
$\mathrm{F}(x)$	0.1	0.2	$3k$	$5k$	$7k$	$10k$

(a) Find the value of the constant k

(1)

(b) Find the probability distribution of X

(3)

A biased die with eight faces is rolled. The discrete random variable Y represents the score which is uppermost. The probability distribution of Y is shown in the table below, where a and b are constants.

y	1	2	3	4	5	6	7	8
$\mathrm{P}(Y = y)$	a	a	a	b	b	b	0.11	0.05

Given that $\mathrm{E}(Y) = 4.02$

(c) form and solve two equations in a and b to show that $a = 0.15$
You must show your working.

(*Solutions relying on calculator technology are not acceptable*.)

(3)

(d) Show that $\mathrm{E}(Y^2) = 20.7$

(2)

(e) Find $\mathrm{Var}(5 - 2Y)$

(3)

These dice are each rolled once. The scores on the two dice are independent.

(f) Find the probability that the sum of these two scores is 3

(2)

1. Jen has one fair 4-sided red die and one fair 4-sided blue die.

- The red die has sides numbered 1, 2, 3 and 4
- The blue die has sides numbered 1, 3, 5 and 7

The discrete random variable R represents the score from one roll of the red die.

The discrete random variable B represents the score from one roll of the blue die.

(a) Write down the name of the distribution of R

(1)

(b) Find $P(R < 3)$

(1)

(c) Write down the value of

(i) $E(R)$

(ii) $E(B)$

(2)

(d) Showing your working, find $\text{Var}(B)$

(3)

Jen rolls each die once.

(e) Find $P(R + B \leqslant 5)$

(2)

(f) Find $P(R^2 < B)$

(3)

The random variable D is defined as the magnitude of the difference between the score on the red die and the score on the blue die.

The table below shows the cumulative distribution function of D

d	0	1	2	3	4	5	6
$F(d)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{16}$	$\frac{3}{4}$	p	$\frac{15}{16}$	1

(g) Showing your working, find the value of p

(3)