P2 Chapter 1 Algebraic Methods

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5. $f(x) = x^3 - 8x + a$

When $f(x)$ is divided by $(x + 2)$ the remainder is -24

(a) Use the remainder theorem to show that $a = -32$

(2)

Given also that $(x - 4)$ is a factor of $f(x)$

(b) (i) write $f(x)$ as a product of two algebraic factors,

(ii) prove that the equation $f(x) = 0$ has only one real root.

(4)

Answer ALL questions. Write your answers in the spaces provided.

1. $f(x) = x^4 + x^3 + 2x^2 + ax + b,$

where a and b are constants.

When $f(x)$ is divided by $(x - 1)$, the remainder is 7

(a) Show that $a + b = 3$

(2)

When $f(x)$ is divided by $(x + 2)$, the remainder is -8

(b) Find the value of a and the value of b

(5)

Given $n \in \mathbb{N}$, prove, by exhaustion, that $n^2 + 2$ is not divisible by 4.

(4)

3. (i) Use algebra to prove that for all real values of x

$$(x-4)^2 \geqslant 2x-9$$

(3)

(ii) Show that the following statement is untrue.

$2^n + 1$ is a prime number for all values of n, $n \in \mathbb{N}$

(1)

6. $f(x) = kx^3 - 15x^2 - 32x - 12$ where k is a constant

Given $(x - 3)$ is a factor of $f(x)$,

(a) show that $k = 9$

(2)

(b) Using algebra and showing each step of your working, fully factorise $f(x)$.

(4)

(c) Solve, for $0 \leqslant \theta < 360°$, the equation

$$9\cos^3\theta - 15\cos^2\theta - 32\cos\theta - 12 = 0$$

giving your answers to one decimal place.

(2)

4. $f(x) = (x - 3)(3x^2 + x + a) - 35$ where a is a constant

(a) State the remainder when $f(x)$ is divided by $(x - 3)$.

(1)

Given $(3x - 2)$ is a factor of $f(x)$,

(b) show that $a = -17$

(2)

(c) Using algebra and showing each step of your working, fully factorise $f(x)$.

(5)

3. $f(x) = 6x^3 + 17x^2 + 4x - 12$

(a) Use the factor theorem to show that $(2x + 3)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, write $f(x)$ as a product of three linear factors.

(4)

(c) Solve, for $\frac{\pi}{2} < \theta < \pi$, the equation

$$6\tan^3\theta + 17\tan^2\theta + 4\tan\theta - 12 = 0$$

giving your answers to 3 significant figures.

(2)

1. $f(x) = x^4 + ax^3 - 3x^2 + bx + 5$

where a and b are constants.

When $f(x)$ is divided by $(x + 1)$, the remainder is 4

(a) Show that $a + b = -1$

(2)

When $f(x)$ is divided by $(x - 2)$, the remainder is -23

(b) Find the value of a and the value of b.

(4)

5. (i) Use algebra to prove that for all $x \geqslant 0$

$$3x + 1 \geqslant 2\sqrt{3x}$$

(3)

(ii) Show that the following statement is not true.

"The sum of three consecutive prime numbers is always a multiple of 5"

(1)

3. (i) Prove that for all single digit prime numbers, p,

$$p^3 + p \text{ is a multiple of } 10$$

(2)

(ii) Show, using algebra, that for $n \in \mathbb{N}$

$$(n + 1)^3 - n^3 \text{ is not a multiple of } 3$$

(3)

4. $f(x) = (x^2 - 2)(2x - 3) - 21$

(a) State the value of the remainder when $f(x)$ is divided by $(2x - 3)$

(1)

(b) Use the factor theorem to show that $(x - 3)$ is a factor of $f(x)$

(2)

(c) Hence,

(i) factorise $f(x)$

(ii) show that the equation $f(x) = 0$ has only one real root.

(5)

9. (a) Prove that for all positive values of x and y,

$$\frac{x + y}{2} \geqslant \sqrt{xy}$$

(3)

(b) Prove by counter-example that this inequality does not hold when x and y are both negative.

(1)

5. $$f(x) = 3x^3 + Ax^2 + Bx - 10$$

where A and B are integers.

Given that

- when $f(x)$ is divided by $(x - 1)$ the remainder is k
- when $f(x)$ is divided by $(x + 1)$ the remainder is $-10k$
- k is a constant

(a) show that

$$11A + 9B = 83$$

(3)

Given also that $(3x - 2)$ is a factor of $f(x)$,

(b) find the value of A and the value of B.

(3)

(c) Hence find the quadratic expression $g(x)$ such that

$$f(x) = (3x - 2)\,g(x)$$

(2)

10. (i) Prove by counter example that the statement

"if p is a prime number then $2p + 1$ is also a prime number"

is not true.

(1)

(ii) Use proof by exhaustion to prove that if n is an integer then

$$5n^2 + n + 12$$

is always even.

(4)

3. (i) Show that the following statement is **false**:

"$(n + 1)^3 - n^3$ is prime for all $n \in \mathbb{N}$"

(2)

(ii) Given that the points $A(1, 0)$, $B(3, -10)$ and $C(7, -6)$ lie on a circle, prove that AB is a diameter of this circle.

(5)

7. $f(x) = Ax^3 + 6x^2 - 4x + B$

where A and B are constants.

Given that

- $(x + 2)$ is a factor of $f(x)$
- $\int_3^5 f(x)\,dx = 176$

find the value of A and the value of B.

(7)

1. Given that a, b and c are integers greater than 0 such that

- $c = b + 2$
- $a + b + c = 10$

Prove, by exhaustion, that the product of a, b and c is always even.

You may use the table below to illustrate your answer.

(3)

You may not need to use all rows of this table.

a	b	c
	1	
	2	

5. $$f(x) = x^3 + (p + 3)x^2 - x + q$$

where p and q are constants and $p > 0$

Given that $(x - 3)$ is a factor of $f(x)$

(a) show that

$$9p + q = -51$$

(2)

Given also that when $f(x)$ is divided by $(x + p)$ the remainder is 9

(b) show that

$$3p^2 + p + q - 9 = 0$$

(2)

(c) Hence find the value of p and the value of q.

(3)

(d) Hence find a quadratic expression $g(x)$ such that

$$f(x) = (x - 3)g(x)$$

(2)

2. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$f(x) = 4x^3 - 8x^2 + 5x + a$$

where a is a constant.

Given that $(2x - 3)$ is a factor of $f(x)$,

(a) use the factor theorem to show that $a = -3$

(2)

(b) Hence show that the equation $f(x) = 0$ has only one real root.

(4)

8. (i) A student writes the following statement:

"When a and b are consecutive **prime** numbers, $a^2 + b^2$ is never a multiple of 10"

Prove by counter example that this statement is **not** true.

(2)

(ii) Given that x and y are even integers greater than 0 and less than 6, prove by exhaustion, that

$$1 < x^2 - \frac{xy}{4} < 15$$

(3)

1. Given that a, b and c are integers greater than 0 such that

- $c = 3a + 1$
- $a + b + c = 15$

prove, by exhaustion, that the product abc is always a multiple of 4

You may use the table below to illustrate your answer.

(3)

You may not need to use all rows of this table.

a	b	c	abc

4. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$f(x) = 4x^3 + ax^2 - 29x + b$$

where a and b are constants.

Given that $(2x + 1)$ is a factor of $f(x)$,

(a) show that

$$a + 4b = -56$$

(2)

Given also that when $f(x)$ is divided by $(x - 2)$ the remainder is -25

(b) find a second simplified equation linking a and b.

(2)

(c) Hence, using algebra and showing your working,

(i) find the value of a and the value of b,

(ii) fully factorise $f(x)$.

(5)

1. $f(x) = ax^3 + 3x^2 - 8x + 2$ where a is a constant

Given that when $f(x)$ is divided by $(x - 2)$ the remainder is 3, find the value of a.

(3)

8. (i) Use a counter example to show that the following statement is **false**

"$n^2 + 3n + 1$ is prime for all $n \in \mathbb{N}$"

(2)

(ii) Use algebra to prove by exhaustion that for all $n \in \mathbb{N}$

"$n^2 - 2$ is **not** a multiple of 4"

(4)

4. $f(x) = (x - 2)(2x^2 + 5x + k) + 21$

where k is a constant.

(a) State the remainder when $f(x)$ is divided by $(x - 2)$

(1)

Given that $(2x - 1)$ is a factor of $f(x)$

(b) show that $k = 11$

(2)

(c) Hence

(i) fully factorise $f(x)$,

(ii) find the number of real solutions of the equation

$$f(x) = 0$$

giving a reason for your answer.

(5)

5. **In this question you must show detailed reasoning.**

(a) Given that x and y are positive numbers such that

$$(x-y)^3 > x^3 - y^3$$

prove that

$$y > x$$

(4)

(b) Using a counter example, show that the result in part (a) is not true for all real numbers.

(2)

6. $f(x) = 4x^3 + px^2 + 8x + q$

where p and q are constants.

Given that

- $(2x + 3)$ is a factor of $f(x)$
- $f(x)$ has a remainder of -5 when divided by $(x + 2)$

(a) (i) show that $p = 10$

(ii) find the value of q.

(5)

(b) Hence find the range of values of x for which $f(x)$ is decreasing.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

3. $f(x) = 2x^3 - x^2 + Ax + B$

where A and B are integers.

Given that when $f(x)$ is divided by $(x + 3)$ the remainder is 55

(a) show that

$$3A - B = -118$$

(2)

Given also that $(2x - 5)$ is a factor of $f(x)$,

(b) find the value of A and the value of B.

(3)

(c) Hence find the quotient when $f(x)$ is divided by $(x - 7)$

(2)

11. (i) Prove by counter example that the statement

"If n is a prime number then $3^n + 2$ is also a prime number."

is false.

(2)

(ii) Use proof by exhaustion to prove that if m is an integer that is **not** divisible by 3, then

$$m^2 - 1$$

is divisible by 3

(4)

5. $f(x) = 3x^3 + ax^2 - 10x + b$

where a and b are constants.

Given that $(3x - 4)$ is a factor of $f(x)$,

(a) show that $16a + 9b = 56$

(2)

Given further that when $f(x)$ is divided by $(x - 2)$ the remainder is b,

(b) find the value of a and the value of b.

(4)

(c) Hence, using algebra, fully factorise $f(x)$.

(3)

8. (i) A student states

"If x and y are irrational numbers, $x \neq y$, then xy is also irrational."

Show, by counter example, that this statement is not always true.

(1)

(ii) Prove, using algebra, that for all odd integers n, the value of the expression

$$n^3 + 3n + 2$$

is always even but never a multiple of 4

(4)