Chapter 3: Vectors in Mechanics

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2. [*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively and position vectors are given relative to a fixed origin O.*]

At time $t = 0$, a bird A leaves its nest, that is located at the point with position vector $(20\mathbf{i} - 17\mathbf{j})$ m, and flies with constant velocity $(-6\mathbf{i} + 7\mathbf{j})\,\text{m s}^{-1}$. At the same time a second bird B leaves its nest which is located at the point with position vector $(-8\mathbf{i} + 9\mathbf{j})$ m and flies with constant velocity $(p\mathbf{i} + 2p\mathbf{j})\,\text{m s}^{-1}$, where p is a constant. At time $t = 4$ s, bird B is south west of bird A.

(a) Find the direction of motion of A, giving your answer as a bearing to the nearest degree.

(3)

(b) Find the speed of B.

(10)

7. [*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively and position vectors are given relative to a fixed origin O.*]

A ship A is moving with constant velocity $(2\mathbf{i} - 14\mathbf{j})\,\text{km h}^{-1}$. At 2 pm the position vector of ship A is $(8\mathbf{i} + 7\mathbf{j})$ km.

A ship B is moving with constant velocity $(12\mathbf{i} - 4\mathbf{j})\,\text{km h}^{-1}$. At 2 pm the position vector of ship B is $(\mathbf{i} + 2\mathbf{j})$ km.

(a) Show that at time t hours after 2 pm,

$$\overrightarrow{BA} = [(7 - 10t)\mathbf{i} + (5 - 10t)\mathbf{j}]\,\text{km}$$

(5)

(b) Hence find the length of time for which the ships are within 2 km of each other.

(7)

8. [*In this question, the horizontal unit vectors* **i** *and* **j** *are directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

Two speedboats, A and B, are each moving with constant velocity. The velocity of A is $20\,\text{km h}^{-1}$ due west and the velocity of B is $40\,\text{km h}^{-1}$ on a bearing of 150°. The boats are modelled as particles.

At noon, the position vector of A is $60\mathbf{i}$ km and B is at the origin O. At time t hours after noon, the position vector of A is $\mathbf{r}$ km and the position vector of B is $\mathbf{s}$ km.

(a) Find the velocity of B in the form $(p\mathbf{i} + q\mathbf{j})\,\text{km h}^{-1}$

(3)

(b) Find expressions for $\mathbf{r}$ and $\mathbf{s}$ in terms of t, $\mathbf{i}$ and $\mathbf{j}$.

(3)

(c) Find the time, to the nearest minute, at which the distance between the boats is the same as it was at noon.

(8)

5. A particle P is moving in a plane with constant acceleration.
The velocity, $\mathbf{v}\,\text{m s}^{-1}$, of P at time t seconds is given by

$$\mathbf{v} = (7 - 5t)\mathbf{i} + (12t - 20)\mathbf{j}$$

(a) Find the speed of P when $t = 2$

(3)

(b) Find, to the nearest degree, the size of the angle between the direction of motion of P and the vector $\mathbf{j}$, when $t = 2$

(3)

The constant acceleration of P is $\mathbf{a}\,\text{m s}^{-2}$

(c) Find $\mathbf{a}$ in terms of $\mathbf{i}$ and $\mathbf{j}$

(3)

(d) Find the value of t when P is moving in the direction of the vector $(-5\mathbf{i} + 8\mathbf{j})$

(4)

6. Two girls, Agatha and Brionie, are roller skating inside a large empty building. The girls are modelled as particles.

At time $t = 0$, Agatha is at the point with position vector $(11\mathbf{i} + 11\mathbf{j})$ m and Brionie is at the point with position vector $(7\mathbf{i} + 16\mathbf{j})$ m. The position vectors are given relative to the door, O, and $\mathbf{i}$ and $\mathbf{j}$ are horizontal perpendicular unit vectors.

Agatha skates with constant velocity $(3\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$

Brionie skates with constant velocity $(4\mathbf{i} - 2\mathbf{j})\,\text{m s}^{-1}$

(a) Find the position vector of Agatha at time t seconds.

(2)

At time $t = 6$ seconds, Agatha passes through the point P.

(b) Show that Brionie also passes through P and find the value of t when this occurs.

(4)

At time t seconds, Agatha is at the point A and Brionie is at the point B.

(c) Show that $\overrightarrow{AB} = [(t - 4)\mathbf{i} + (5 - t)\mathbf{j}]$ m

(2)

(d) Find the distance between the two girls when they are closest together.

(4)

5. [*In this question* **i** *and* **j** *are perpendicular horizontal unit vectors.*]

A particle P is moving with constant acceleration. At 2 pm, the velocity of P is $(3\mathbf{i} + 5\mathbf{j})$ km h^{-1} and at 2.30 pm the velocity of P is $(\mathbf{i} + 7\mathbf{j})$ km h^{-1}

At time T hours after 2 pm, P is moving in the direction of the vector $(-\mathbf{i} + 2\mathbf{j})$

(a) Find the value of T.

(6)

Another particle, Q, has velocity $\mathbf{v}_Q$ km h^{-1} at time t hours after 2 pm, where

$$\mathbf{v}_Q = (-4 - 2t)\mathbf{i} + (\mu + 3t)\mathbf{j}$$

and μ is a constant.

Given that there is an instant when the velocity of P is equal to the velocity of Q,

(b) find the value of μ.

(3)

4. The position vector, **r** metres, of a particle P at time t seconds, relative to a fixed origin O, is given by

$$\mathbf{r} = (t - 3)\mathbf{i} + (1 - 2t)\mathbf{j}$$

(a) Find, to the nearest degree, the size of the angle between **r** and the vector **j**, when $t = 2$

(3)

(b) Find the values of t for which the distance of P from O is 2.5 m.

(5)

8. [*In this question* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin.*]

At 7 am a ship leaves a port and moves with constant velocity. The position vector of the port is $(-2\mathbf{i} + 9\mathbf{j})$ km.

At 7.36 am the ship is at the point with position vector $(4\mathbf{i} + 6\mathbf{j})$ km.

(a) Show that the velocity of the ship is $(10\mathbf{i} - 5\mathbf{j})\,\text{km h}^{-1}$

(2)

(b) Find the position vector of the ship t hours after leaving port.

(2)

At 8.48 am a passenger on the ship notices that a lighthouse is due east of the ship.

At 9 am the same passenger notices that the lighthouse is now north east of the ship.

(c) Find the position vector of the lighthouse.

(4)

(d) Find the position vector of the ship when it is due south of the lighthouse.

(4)

6. [*In this question* **i** *and* **j** *are horizontal unit vectors*.]

A particle P of mass 2 kg moves under the action of two forces, $(p\mathbf{i} + q\mathbf{j})$ N and $(2q\mathbf{i} + p\mathbf{j})$ N, where p and q are constants.

Given that the acceleration of P is $(\mathbf{i} - \mathbf{j})\,\text{m s}^{-2}$

(a) find the value of p and the value of q.

(5)

(b) Find the size of the angle between the direction of the acceleration and the vector **j**.

(2)

At time $t = 0$, the velocity of P is $(3\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$

At $t = T$ seconds, P is moving in the direction of the vector $(11\mathbf{i} - 13\mathbf{j})$.

(c) Find the value of T.

(5)

8. [*In this question* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin.*]

A ship A moves with constant velocity $(3\mathbf{i} - 10\mathbf{j})\,\text{km h}^{-1}$

At time t hours, the position vector of A is $\mathbf{r}$ km.

At time $t = 0$, A is at the point with position vector $(13\mathbf{i} + 5\mathbf{j})$ km.

(a) Find $\mathbf{r}$ in terms of t.

(2)

Another ship B moves with constant velocity $(15\mathbf{i} + 14\mathbf{j})\,\text{km h}^{-1}$

At time $t = 0$, B is at the point with position vector $(3\mathbf{i} - 5\mathbf{j})$ km.

(b) Show that, at time t hours,

$$\overrightarrow{AB} = \left[(12t - 10)\mathbf{i} + (24t - 10)\mathbf{j}\right]\text{km}$$

(4)

Given that the two ships do not change course,

(c) find the shortest distance between the two ships,

(6)

(d) find the bearing of ship B from ship A when the ships are closest.

(2)

6. A particle P is moving with constant acceleration.

At time $t = 1$ second, P has velocity $(-\mathbf{i} + 4\mathbf{j})\,\mathrm{m\,s^{-1}}$

At time $t = 4$ seconds, P has velocity $(5\mathbf{i} - 8\mathbf{j})\,\mathrm{m\,s^{-1}}$

Find the speed of P at time $t = 3.5$ seconds.

(6)

8. [*In this question,* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

Two boats, P and Q, are moving with constant velocities.

The velocity of P is $15\mathbf{i}\,\text{m s}^{-1}$ and the velocity of Q is $(20\mathbf{i} - 20\mathbf{j})\,\text{m s}^{-1}$

(a) Find the direction in which Q is travelling, giving your answer as a bearing.

(2)

The boats are modelled as particles.

At time $t = 0$, P is at the origin O and Q is at the point with position vector $200\mathbf{j}$ m.

At time t seconds, the position vector of P is $\mathbf{p}$ m and the position vector of Q is $\mathbf{q}$ m.

(b) Show that

$$\overrightarrow{PQ} = [5t\mathbf{i} + (200 - 20t)\mathbf{j}]\,\text{m}$$

(5)

(c) Find the bearing of P from Q when $t = 10$

(2)

(d) Find the distance between P and Q when Q is north east of P

(5)

(e) Find the times when P and Q are 200 m apart.

(3)

8. [*In this question,* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

Two ships, A and B, are moving with constant velocities.
The velocity of A is $(3\mathbf{i} + 12\mathbf{j})\,\text{km h}^{-1}$ and the velocity of B is $(p\mathbf{i} + q\mathbf{j})\,\text{km h}^{-1}$

(a) Find the speed of A.

(2)

The ships are modelled as particles.
At 12 noon, A is at the point with position vector $(-9\mathbf{i} + 6\mathbf{j})$ km and B is at the point with position vector $(16\mathbf{i} + 6\mathbf{j})$ km.

At time t hours after 12 noon,

$$\overrightarrow{AB} = [(25 - 12t)\mathbf{i} - 9t\mathbf{j}]\,\text{km}$$

(b) Find the value of p and the value of q.

(7)

(c) Find the bearing of A from B when the ships are 15 km apart, giving your answer to the nearest degree.

(7)

3. A particle P is moving with constant acceleration $(-4\mathbf{i} + \mathbf{j})\,\text{m s}^{-2}$

At time $t = 0$, P has velocity $(14\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$

(a) Find the speed of P at time $t = 2$ seconds.

(3)

(b) Find the size of the angle between the direction of $\mathbf{i}$ and the direction of motion of P at time $t = 2$ seconds.

(3)

At time $t = T$ seconds, P is moving in the direction of vector $(2\mathbf{i} - 3\mathbf{j})$

(c) Find the value of T

(4)

8.

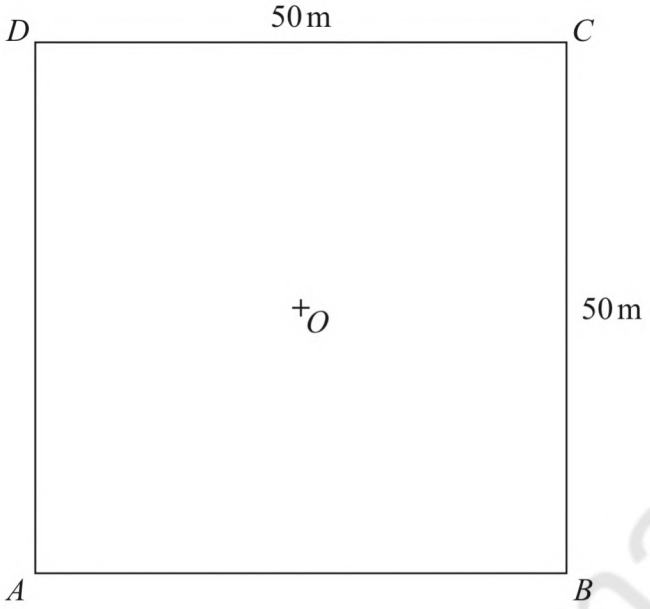

Figure 5

A square floor space $ABCD$, with centre O, is modelled as a flat horizontal surface measuring 50 m by 50 m, as shown in Figure 5.
The horizontal unit vectors **i** and **j** are in the direction of $\overrightarrow{AB}$ and $\overrightarrow{AD}$ respectively.

All position vectors are given relative to O.

A small robot R is programmed to travel across the floor at a constant velocity.

- At time $t = 0$, R is at the point with position vector $(-2\mathbf{i} + \mathbf{j})$ m
- At time $t = 11$ s, R is at the point with position vector $(9\mathbf{i} + 23\mathbf{j})$ m
- At time t seconds, the position vector of R is **r** metres

(a) Find, in terms of t, **i** and **j**, an expression for **r**

(3)

A second robot S is at the point C.

- At time $t = 0$, S leaves C and moves with constant velocity $(-\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$
- At time t seconds, the position vector of S is **s** metres

(b) Write down, in terms of t, **i** and **j**, an expression for **s**

(1)

(c) Show that

$$\overrightarrow{SR} = [(2t - 27)\mathbf{i} + (3t - 24)\mathbf{j}]\text{ m}$$

(2)

(d) Find the time when the distance between R and S is a minimum.

(3)

4. [*In this question* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively.*]

A particle P moves with constant acceleration $(-\lambda\mathbf{i} + 2\lambda\mathbf{j})\,\text{m s}^{-2}$, where λ is a positive constant.

At time $t = 0$, the velocity of P is $(5\mathbf{i} - 8\mathbf{j})\,\text{m s}^{-1}$

(a) Find the velocity of P when $t = 5\,\text{s}$, giving your answer in terms of **i**, **j** and λ.

(2)

The speed of P when $t = 5\,\text{s}$ is $13\,\text{m s}^{-1}$

(b) Show that

$$25\lambda^2 - 42\lambda - 16 = 0$$

(3)

(c) Find the direction of motion of P when $t = 4\,\text{s}$, giving your answer as a bearing to the nearest degree.

(5)

6. [*In this question* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

At 12:00, a ship P sets sail from a harbour with position vector $(15\mathbf{i} + 36\mathbf{j})$ km.
At 12:30, P is at the point with position vector $(20\mathbf{i} + 34\mathbf{j})$ km.

Given that P moves with constant velocity,

(a) show that the velocity of P is $(10\mathbf{i} - 4\mathbf{j})\,\text{km h}^{-1}$

(2)

At time t hours after 12:00, the position vector of P is $\mathbf{p}$ km.

(b) Find an expression for $\mathbf{p}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and t.

(2)

A second ship Q is also travelling at a constant velocity.

At time t hours after 12:00, the position vector of Q is given by $\mathbf{q}$ km, where

$$\mathbf{q} = (42 - 8t)\mathbf{i} + (9 + 14t)\mathbf{j}$$

Ships P and Q are modelled as particles.

If both ships maintained their course,

(c) (i) verify that they would collide at 13:30

(ii) find the position vector of the point at which the collision would occur.

(4)

At 12:30 Q changes speed and direction to avoid the collision.

Ship Q now travels due north with a constant speed of $15\,\text{km h}^{-1}$

Ship P maintains the same constant velocity throughout.

(d) Find the exact distance between P and Q at 14:30

(7)

7. [*In this question* **i** *and* **j** *are horizontal unit vectors directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

At midnight, a ship S is at the point with position vector $(19\mathbf{i} + 22\mathbf{j})$ km

The ship travels with constant velocity $(12\mathbf{i} - 16\mathbf{j})\,\text{km h}^{-1}$

(a) Find the speed of S.

(2)

At time t hours after midnight, the position vector of S is $\mathbf{s}$ km.

(b) Find an expression for $\mathbf{s}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and t.

(2)

A lighthouse stands on a small rocky island. The lighthouse is modelled as being at the point with position vector $(26\mathbf{i} + 15\mathbf{j})$ km.

It is not safe for ships to be within 1.3 km of the lighthouse.

(c) (i) Find the value of t when S is closest to the lighthouse.

(ii) Hence determine whether it is safe for S to continue its course.

(7)

6. A force $\mathbf{F}$ is given by $\mathbf{F} = (10\mathbf{i} + \mathbf{j})$ N.

(a) Find the exact value of the magnitude of $\mathbf{F}$.

(2)

(b) Find, in degrees, the size of the angle between the direction of $\mathbf{F}$ and the direction of the vector $(\mathbf{i} + \mathbf{j})$.

(4)

The resultant of the force $\mathbf{F}$ and the force $(-15\mathbf{i} + a\mathbf{j})$ N, where a is a constant, is parallel to, but in the opposite direction to, the vector $(2\mathbf{i} - 3\mathbf{j})$.

(c) Find the value of a.

(5)

7. [*In this question, the horizontal unit vectors* **i** *and* **j** *are directed due east and due north respectively and position vectors are given relative to a fixed origin O.*]

Two speedboats, A and B, are each moving with constant velocity.

- the velocity of A is $40\,\text{km h}^{-1}$ due east
- the velocity of B is $20\,\text{km h}^{-1}$ on a bearing of angle α ($0° < \alpha < 90°$), where $\tan\alpha = \frac{4}{3}$

The boats are modelled as particles.

(a) Find, in terms of **i** and **j**, the velocity of B in km h^{-1}

(2)

At noon

- the position vector of A is $20\mathbf{j}\,\text{km}$
- the position vector of B is $(10\mathbf{i} + 5\mathbf{j})\,\text{km}$

At time t hours after noon

- the position vector of A is $\mathbf{r}\,\text{km}$, where $\mathbf{r} = 20\mathbf{j} + 40t\mathbf{i}$
- the position vector of B is $\mathbf{s}\,\text{km}$

(b) Find an expression for $\mathbf{s}$ in terms of t, **i** and **j**.

(2)

(c) Show that at time t hours after noon,

$$\overrightarrow{AB} = [(10 - 24t)\mathbf{i} + (12t - 15)\mathbf{j}]\,\text{km}$$

(2)

(d) Show that the boats will never collide.

(3)

(e) Find the distance between the boats when the bearing of B from A is 225°

(4)

3. [*In this question* **i** *and* **j** *are horizontal unit vectors and position vectors are given relative to a fixed origin.*]

A ship A is moving with constant velocity.

At 1 pm, the position vector of A is $(25\mathbf{i} + 10\mathbf{j})$ km.

At 3 pm, the position vector of A is $(55\mathbf{i} + 34\mathbf{j})$ km.

At time t hours after 1 pm, the position vector of A is $\mathbf{r}_A$ km.

(a) Show that $\mathbf{r}_A = (25 + 15t)\mathbf{i} + (10 + 12t)\mathbf{j}$

(4)

The speed of A is $V\,\text{m s}^{-1}$

(b) Find the value of V.

(2)

A ship B is moving with constant velocity $(20\mathbf{i} - 6\mathbf{j})\,\text{km h}^{-1}$

At 1 pm, the position vector of B is $(35\mathbf{i} + 51\mathbf{j})$ km.

At 2:30 pm, B passes through the point P.

(c) Show that A also passes through P.

(5)

5. [*In this question* **i** *and* **j** *are horizontal perpendicular unit vectors and position vectors are given relative to a fixed origin*.]

In a game, a ball B is rolled across a horizontal surface towards a fixed target.
The ball is modelled as a particle moving with constant velocity.

At time $t = 1$ s, the position vector of B is $(-2\mathbf{i} + 5\mathbf{j})$ m.
At time $t = 9$ s, the position vector of B is $(10\mathbf{i} - 3\mathbf{j})$ m.

(a) Find the velocity of the ball.

(3)

The position vector of the target is $(13\mathbf{i} - 5\mathbf{j})$ m.

(b) Use the model to find the distance of B from the target at time $t = 7$ s.

(4)