Chapter 3: Centre of Mass

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M2 Chapter 3 Solutions

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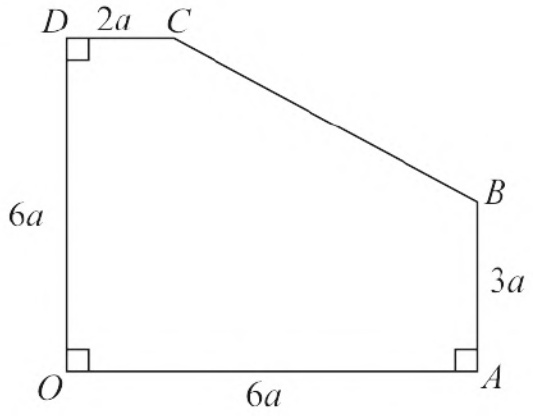

Figure 1

The uniform lamina $OABCD$ is shown in Figure 1, with $OA = 6a$, $AB = 3a$, $CD = 2a$ and $DO = 6a$ and with right angles at O, A and D.

(a) Find the distance of the centre of mass of the lamina

(i) from OD,

(ii) from OA.

(6)

The lamina is suspended from C and hangs freely in equilibrium with CB inclined at an angle α to the vertical.

(b) Find, to the nearest degree, the size of the angle α.

(4)

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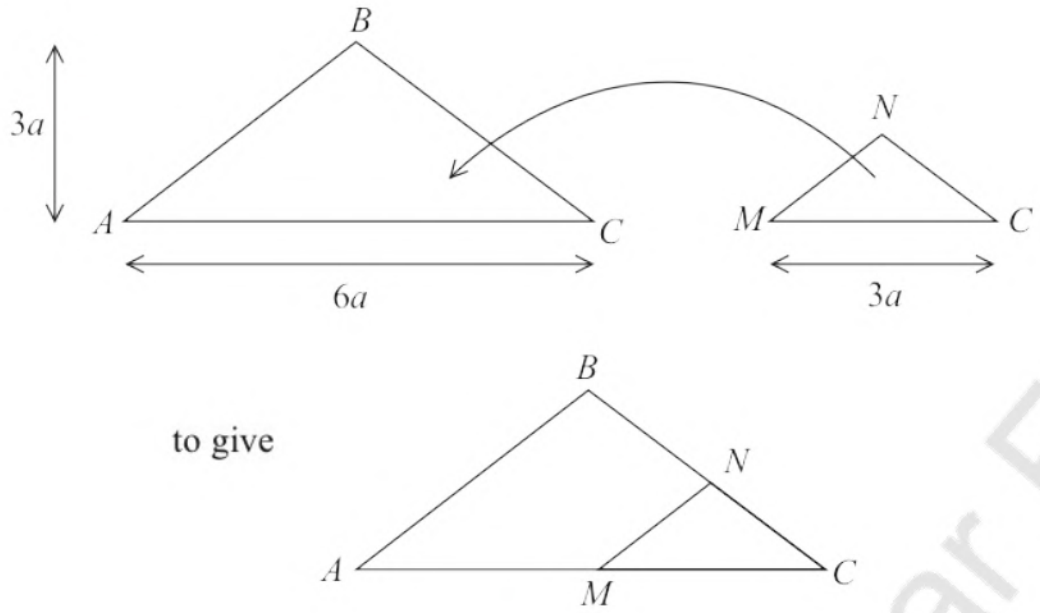

Figure 2

The uniform lamina ABC is an isosceles triangle with $AB = BC$, $AC = 6a$ and the distance from B to AC is $3a$.

The uniform lamina MNC is an isosceles triangle with $MN = NC$ and $MC = 3a$. Triangles ABC and MNC are similar and are made of the same material.

The lamina L is formed by fixing triangle MNC on top of triangle ABC, as shown in Figure 2.

(a) Show that the distance of the centre of mass of L from AC is $\frac{9}{10}a$

(5)

The lamina L is freely suspended from B and hangs in equilibrium.

(b) Find, to the nearest degree, the size of the angle between AB and the downward vertical.

(7)

5.

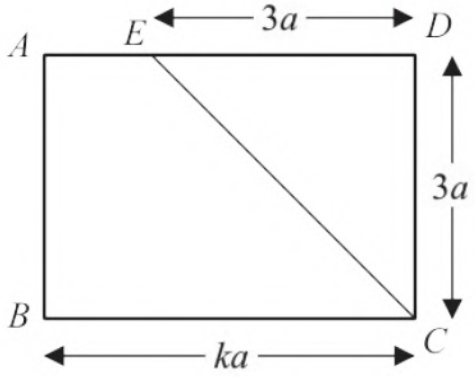

Figure 3

Figure 3 shows a uniform rectangular lamina $ABCD$ with sides of length $3a$ and ka, where $k > 3$. The point E on side AD is such that $DE = 3a$. Rectangle $ABCD$ is folded along the line CE to produce the folded lamina L shown in Figure 4.

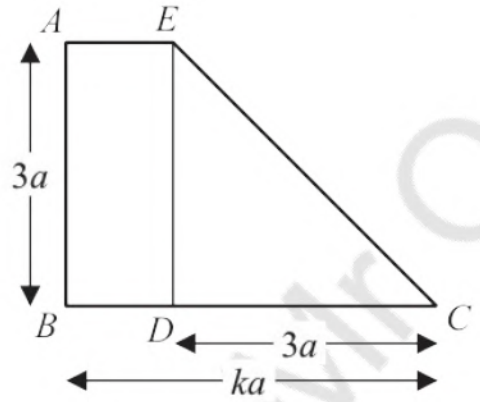

Figure 4

Find, in terms of a and k,

(a) the distance of the centre of mass of L from AB,

(5)

(b) the distance of the centre of mass of L from AE.

(4)

The folded lamina L is freely suspended from A and hangs in equilibrium with AB at 45° to the downward vertical.

(c) Find, to 3 significant figures, the value of k.

(4)

1. Three particles of masses $3m$, m and $2m$ are positioned at the points with coordinates $(a, 8)$, $(-4, 0)$ and $(5, -2)$ respectively.

Given that the centre of mass of the three particles is at the point with coordinates $(k, 2k)$, where k is a constant, find the value of a.

(5)

4.

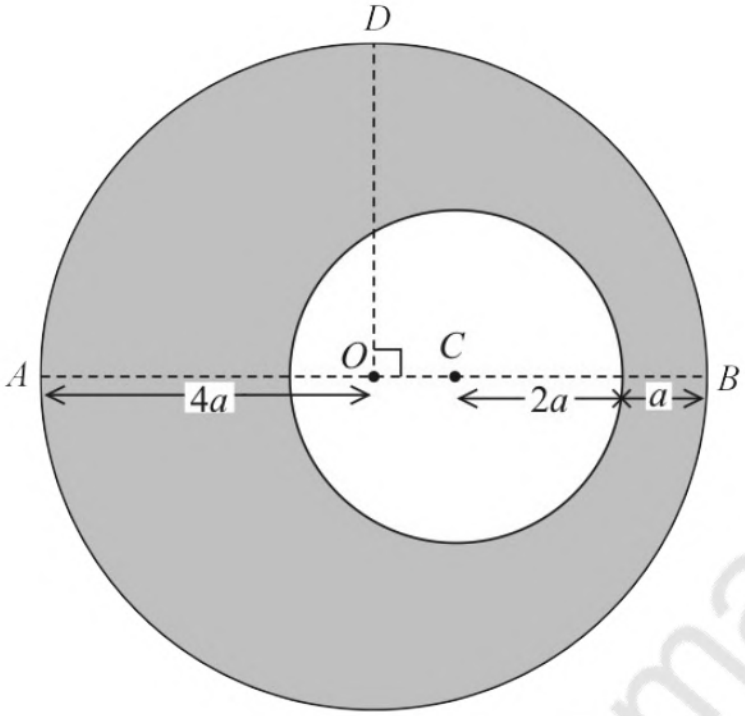

Figure 1

The uniform lamina L, shown shaded in Figure 1, is formed by removing a circular disc of radius $2a$ from a uniform circular disc of radius $4a$. The larger disc has centre O and diameter AB. The radius OD is perpendicular to AB. The smaller disc has centre C, where C is on AB and $BC = 3a$

(a) Show that the centre of mass of L is $\frac{13}{3}a$ from B.

(4)

The mass of L is M and a particle of mass kM is attached to L at B. When L, with the particle attached, is freely suspended from point D, it hangs in equilibrium with A higher than B and AB at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$

(b) Find the value of k.

(5)

4. [*The centre of mass of a uniform semicircular lamina of radius r is* $\frac{4r}{3\pi}$ *from the centre.*]

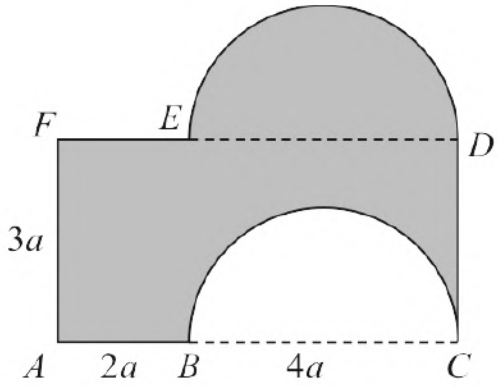

Figure 3

The uniform rectangular lamina $ABCDEF$ has sides $AC = FD = 6a$ and $AF = CD = 3a$. The point B lies on AC with $AB = 2a$ and the point E lies on FD with $FE = 2a$.

The template, T, shown shaded in Figure 3, is formed by removing the semicircular lamina with diameter BC from the rectangular lamina and then fixing this semicircular lamina to the opposite side, FD, of the rectangular lamina. The diameter of the semicircular lamina coincides with ED and the semicircular arc ED is outside the rectangle $ABCDEF$. All points of T lie in the same plane.

(a) Show that the centre of mass of T is a distance $\left(\frac{9 + 2\pi}{6}\right)a$ from AC.

(4)

The mass of T is M. A particle of mass kM is attached to T at C. The loaded template is freely suspended from A and hangs in equilibrium with AF at angle ϕ to the downward vertical through A.

Given that $\tan\phi = \frac{3}{2}$

(b) find the value of k.

(6)

4.

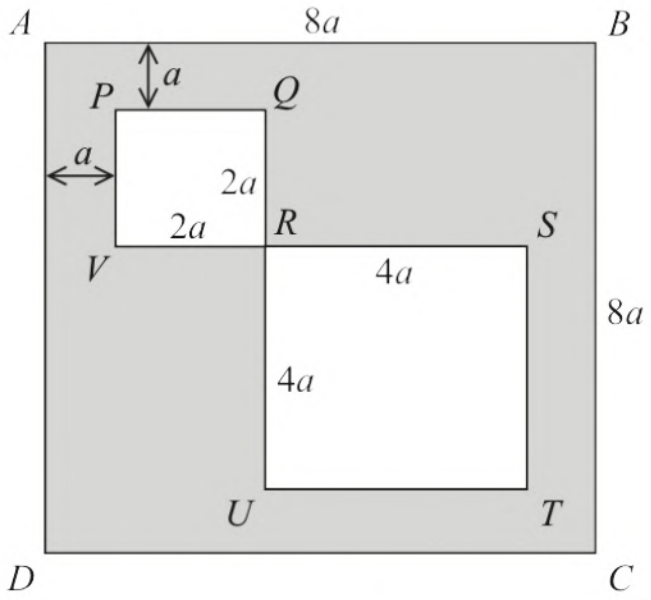

Figure 2

The uniform lamina L, shown shaded in Figure 2, is formed by removing the square $PQRV$, of side $2a$, and the square $RSTU$, of side $4a$, from a uniform square lamina $ABCD$, of side $8a$. The lines QRU and VRS are straight. The side AD is parallel to PV and the side AB is parallel to PQ. The distance between AD and PV is a and the distance between AB and PQ is a. The centre of mass of L is at the point G.

(a) Show that the distance of G from the side AD is $\frac{42}{11}a$

(5)

The mass of L is M. A particle of mass kM is attached to L at C.

The lamina, with the attached particle, is freely suspended from B and hangs in equilibrium with BC making an angle of $45°$ with the horizontal.

(b) Find the value of k.

(4)

2.

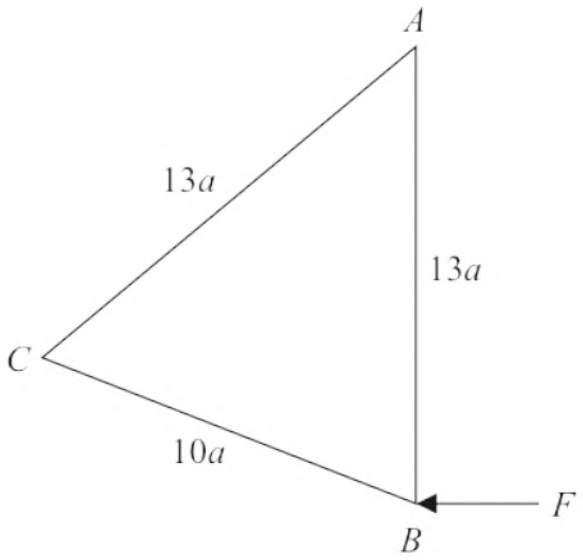

Figure 1

The uniform lamina ABC has sides $AB = AC = 13a$ and $BC = 10a$. The lamina is freely suspended from A. A horizontal force of magnitude F is applied to the lamina at B, as shown in Figure 1. The line of action of the force lies in the vertical plane containing the lamina. The lamina is in equilibrium with AB vertical. The weight of the lamina is W.

Find F in terms of W.

(5)

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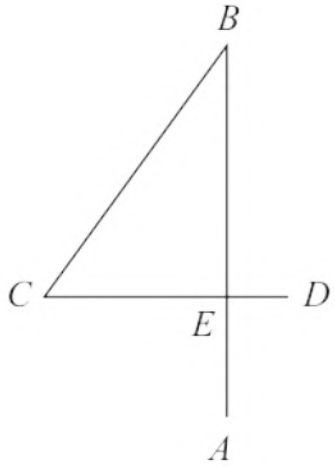

Figure 2

The number "4", shown in Figure 2, is a rigid framework made from three uniform rods, AB, BC and CD, where

$$AB = 6a,\ BC = 5a \text{ and } CD = 4a$$

The point E is on AB and CD, where $BE = 4a$, $CE = 3a$ and angle $CEB = 90°$

The three rods are all made from the same material and they all lie in the same plane.

The framework is suspended from B and hangs in equilibrium with BA at an angle θ to the downward vertical.

Find θ to the nearest degree.

(9)

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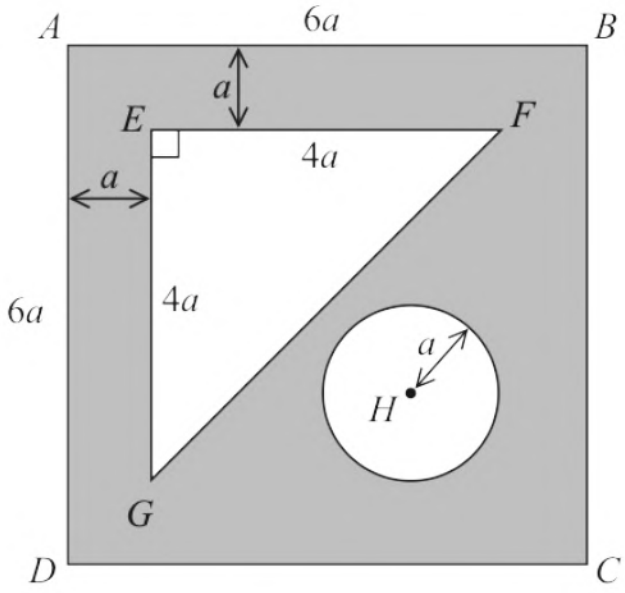

Figure 1

The uniform lamina $ABCD$ is a square of side $6a$. The template T, shown shaded in Figure 1, is formed by removing the right-angled triangle EFG and the circle, centre H and radius a, from the square lamina.

Triangle EFG has $EF = EG = 4a$, with EF parallel to AB and EG parallel to AD. The distance between AB and EF is a and the distance between AD and EG is a.

The point H lies on AC and the distance of H from BC is $2a$.

(a) Show that the centre of mass of T is a distance $\dfrac{4(67 - 3\pi)}{3(28 - \pi)}a$ from AD.

(5)

The template T is suspended from the ceiling by two light inextensible vertical strings. One string is attached to T at A and the other string is attached to T at B so that T hangs in equilibrium with AB horizontal.

The weight of T is W. The tension in the string attached to T at B is kW, where k is a constant.

(b) Find the value of k, giving your answer to 2 decimal places.

(3)

7. In this question you may use, without proof, the formula for the centre of mass of a uniform sector of a circle, as given in the formulae book.

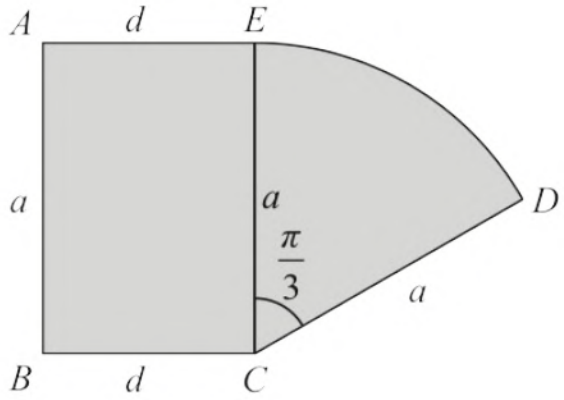

Figure 3

The uniform lamina $ABCDE$, shown shaded in Figure 3, is formed by joining a rectangle to a sector of a circle.

- The rectangle $ABCE$ has $AB = EC = a$ and $AE = BC = d$
- The sector CDE has centre C and radius a
- Angle $ECD = \frac{\pi}{3}$ radians

The centre of mass of the lamina lies on EC.

(a) Show that $a = \sqrt{3}d$

(4)

The lamina is freely suspended from B and hangs in equilibrium with BC at an angle β radians to the downward vertical.

(b) Find the value of β

(7)

6. [*The centre of mass of a semicircular arc of radius r is* $\frac{2r}{\pi}$ *from the centre.*]

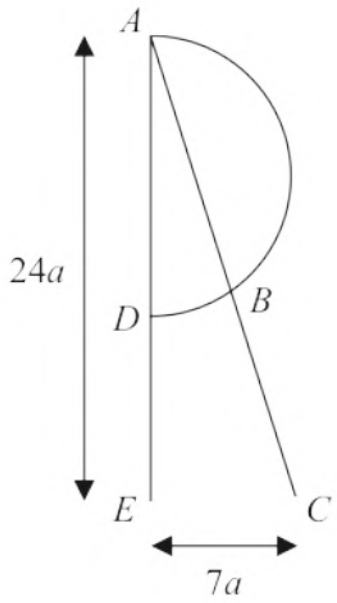

Figure 2

Uniform wire is used to form the framework shown in Figure 2.

In the framework,

- ABC is straight and has length $25a$
- ADE is straight and has length $24a$
- ABD is a semicircular arc of radius $7a$
- $EC = 7a$
- angle $AEC = 90°$
- the points A, B, C, D and E all lie in the same plane

The distance of the centre of mass of the framework from AE is d.

(a) Show that $d = \frac{53}{2(7+\pi)}a$

(4)

The framework is freely suspended from A and hangs in equilibrium with AC at angle $\alpha°$ to the downward vertical.

(b) Find the value of α.

(7)

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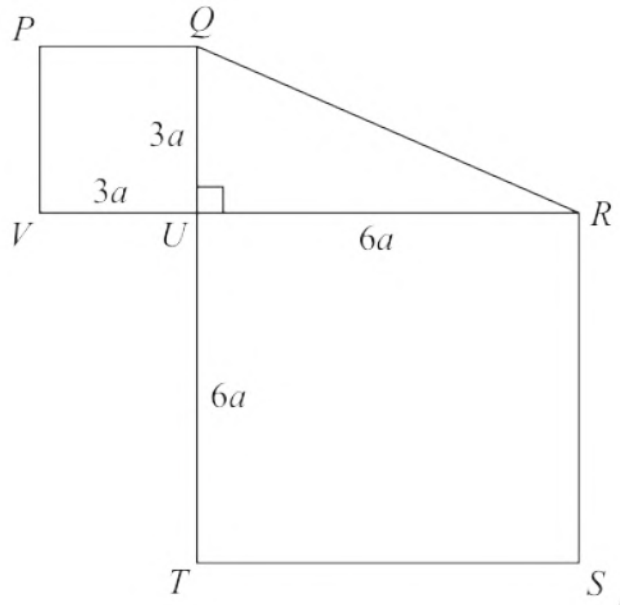

Figure 3

The template shown in Figure 3 is formed by joining together three separate laminas. All three laminas lie in the same plane.

- $PQUV$ is a uniform square lamina with sides of length $3a$
- $URST$ is a uniform square lamina with sides of length $6a$
- QRU is a uniform triangular lamina with $UQ = 3a$, $UR = 6a$ and angle $QUR = 90°$

The mass per unit area of $PQUV$ is k, where k is a constant.
The mass per unit area of $URST$ is k.
The mass per unit area of QRU is $2k$.

The distance of the centre of mass of the template from QT is d.

(a) Show that $d = \frac{29}{14}a$

(5)

The template is freely suspended from the point Q and hangs in equilibrium with QR at $\theta°$ to the downward vertical.

(b) Find the value of θ

(7)

1. Three particles of masses $2m$, $3m$ and $4m$ are placed at the points with coordinates $(-2, 5)$, $(2, -3)$ and $(3k, k)$ respectively, where k is a constant.
The centre of mass of the three particles is at the point $(\bar{x}, \bar{y})$.

(a) Show that $\bar{x} = \dfrac{2 + 12k}{9}$

(2)

The centre of mass of the three particles lies at a point on the straight line with equation $x + 2y = 3$

(b) Find the value of k.

(4)

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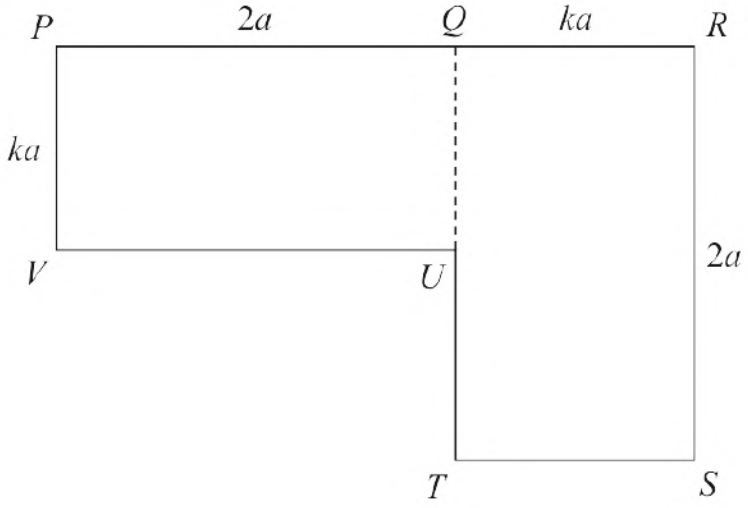

Figure 2

The uniform lamina $PQRSTUV$ shown in Figure 2 is formed from two identical rectangles, $PQUV$ and $QRSTU$.
The rectangles have sides $PQ = RS = 2a$ and $PV = QR = ka$.

(a) Show that the centre of mass of the lamina is $\left(\frac{6+k}{4}\right)a$ from PV

(3)

The lamina is freely suspended from P and hangs in equilibrium with PR at an angle of α to the downward vertical.

Given that $\tan\alpha = \frac{7}{15}$

(b) find the value of k.

(6)

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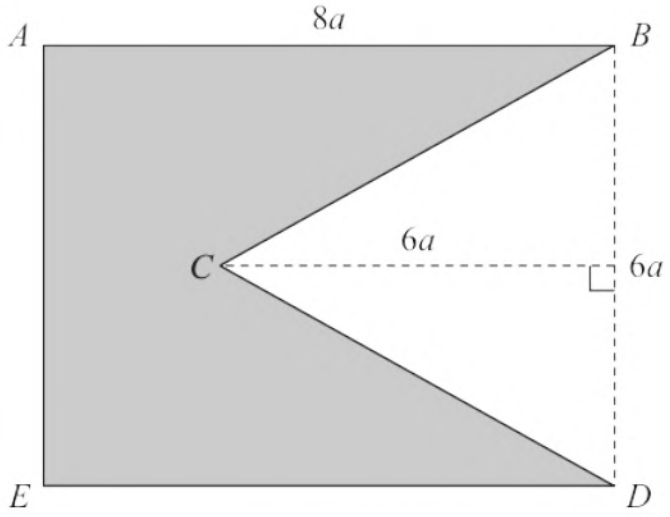

Figure 1

The uniform lamina $ABDE$ is in the shape of a rectangle with $AB = 8a$ and $BD = 6a$.
The triangle BCD is isosceles and has base $6a$ and perpendicular height $6a$.
The template $ABCDE$, shown shaded in Figure 1, is formed by removing the triangular lamina BCD from the lamina $ABDE$.

(a) Show that the centre of mass of the template is $\frac{14}{5}a$ from AE.

(5)

The template is freely suspended from A and hangs in equilibrium with AB at an angle of $\theta°$ to the downward vertical.

(b) Find the value of θ, giving your answer to the nearest whole number.

(3)

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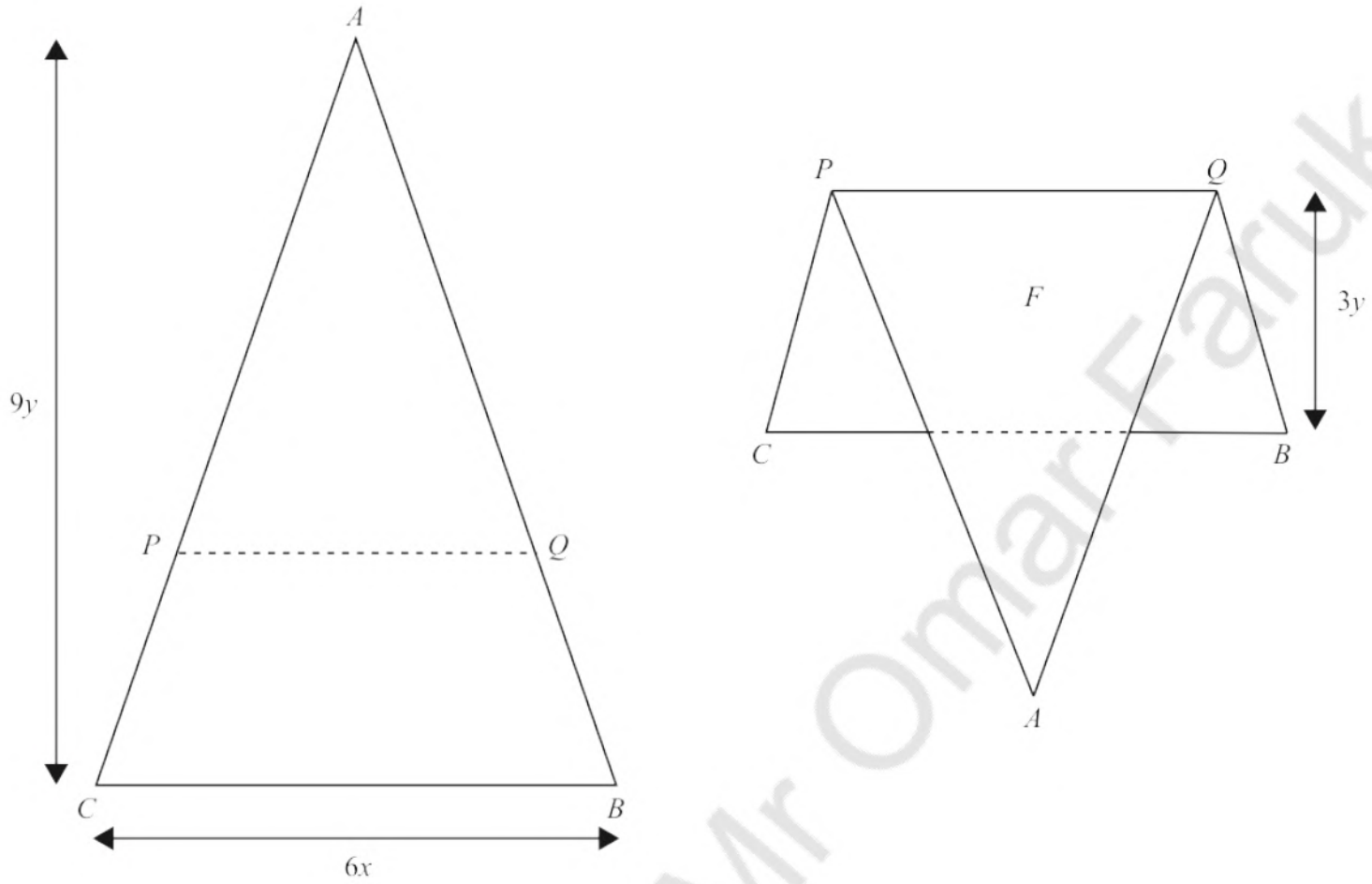

Figure 1

The uniform triangular lamina ABC, shown in Figure 1, has height $9y$, base $BC = 6x$, and $AB = AC$

The points P and Q are such that $AP : PC = AQ : QB = 2 : 1$

The lamina is folded along PQ to form the folded lamina F

The distance of the centre of mass of F from PQ is d

(a) Show that $d = \frac{16}{9}y$

(5)

The folded lamina is suspended from P and hangs freely in equilibrium with PQ at an angle α to the downward vertical.

Given that $\tan\alpha = \frac{64}{81}$

(b) find x in terms of y

(3)

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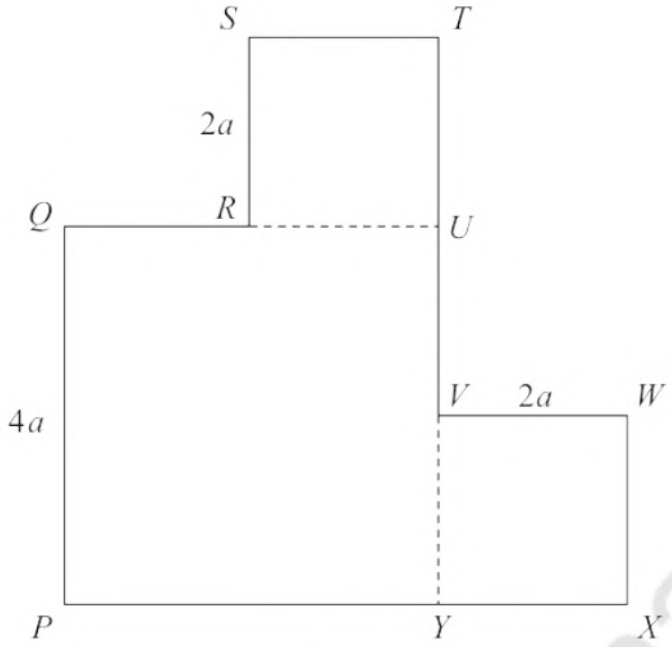

Figure 1

Figure 1 shows a template where

- $PQUY$ is a uniform square lamina with sides of length $4a$
- $RSTU$ is a uniform square lamina with sides of length $2a$
- $VWXY$ is a uniform square lamina with sides of length $2a$
- the three squares all lie in the same plane
- the mass per unit area of $VWXY$ is **double** the mass per unit area of $PQUY$
- the mass per unit area of $RSTU$ is **double** the mass per unit area of $PQUY$
- the distance of the centre of mass of the template from PX is d

(a) Show that $d = \frac{5}{2}a$

(5)

The template is freely pivoted about Q and hangs in equilibrium with PQ at an angle of θ to the downward vertical.

(b) Find the value of $\tan\theta$

(6)

The mass of the template is M

The template is still freely pivoted about Q, but it is now held in equilibrium, with PQ vertical, by a horizontal force of magnitude F which acts on the template at X.
The line of action of the force lies in the same plane as the template.

(c) Find F in terms of M and g

(3)

4.

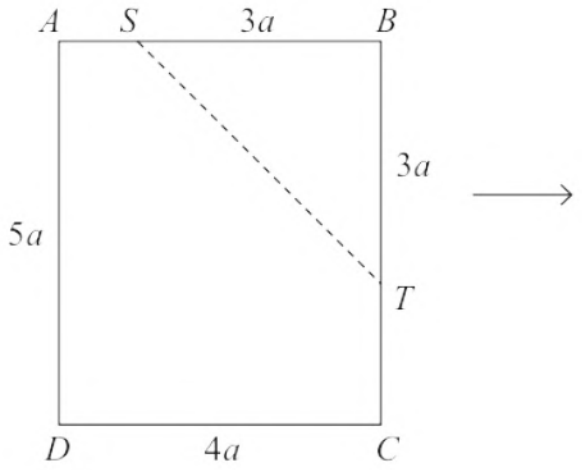

Figure 2

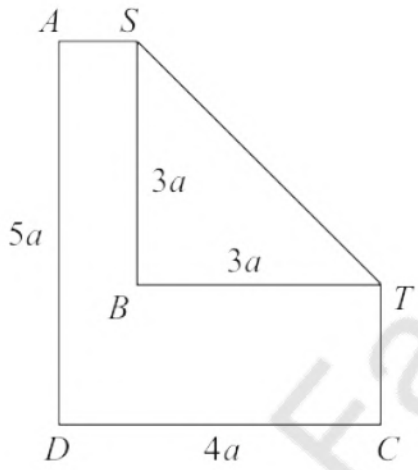

Figure 3

The uniform rectangular lamina $ABCD$, shown in Figure 2, has $DC = 4a$ and $AD = 5a$

The points S on AB and T on BC are such that $SB = BT = 3a$

The lamina is folded along ST to form the folded lamina L, shown in Figure 3.

The distance of the centre of mass of L from AD is d.

(a) Show that $d = \frac{71}{40}a$

(5)

The weight of L is $4W$. A particle of weight W is attached to L at C.

The folded lamina L is freely suspended from S.

A force of magnitude F, acting parallel to DC, is applied to L at D so that AD is vertical.

(b) Find F in terms of W

(4)

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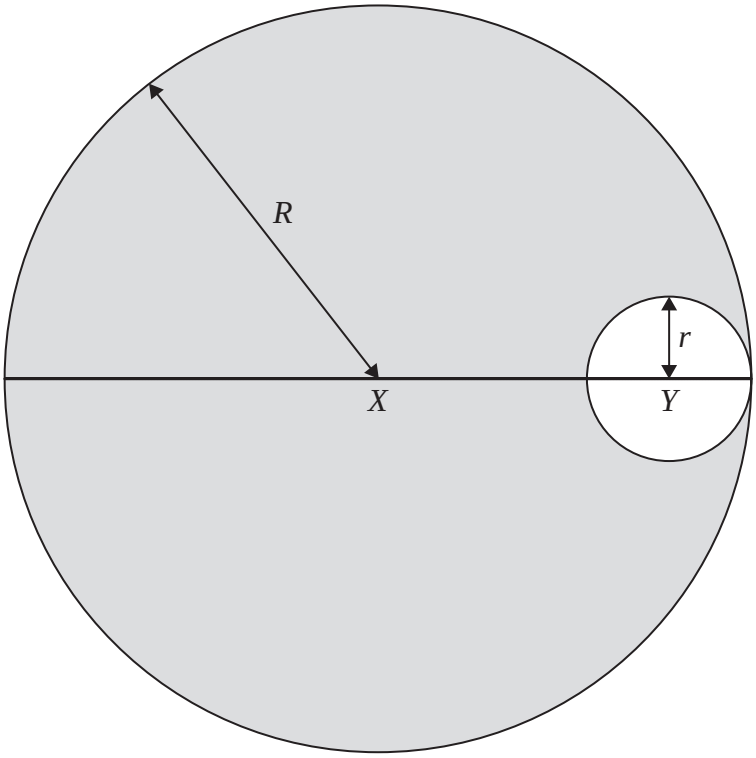

Figure 1

A uniform circular disc C has centre X and radius R.

A disc with centre Y and radius r, where $0 < r < R$ and $XY = R - r$, is removed from C to form the template shown shaded in Figure 1.

The centre of mass of the template is a distance kr from X.

(a) Show that $r = \frac{k}{1-k}R$

(4)

(b) Hence find the range of possible values of k.

(2)

The point P is on the outer edge of the template and PX is perpendicular to XY.

The template is freely suspended from P and hangs in equilibrium.

Given that $k = \frac{4}{9}$

(c) find the angle that XY makes with the vertical.

(3)

The mass of the template is M.

(d) Find, in terms of M, the mass of the lightest particle that could be attached to the template so that it would hang in equilibrium from P with XY horizontal.

(3)