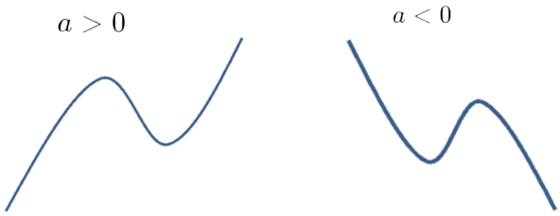
Graphs and Transformations

Cubic Graphs

$$y = ax^3 + bx^2 + cx + d$$

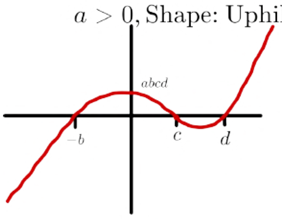


$y = a(x+b)(x-c)(x-d)$

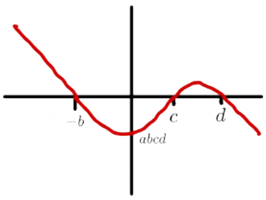
Roots : $x = -b, c, d$

y-intercept : $ab(-c)(-d) = abcd$

$a > 0$, Shape: Uphill



$a < 0$, Shape: Downhill

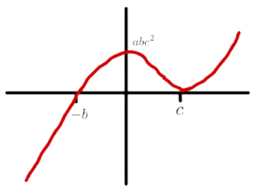


$y = a(x+b)(x-c)^2$

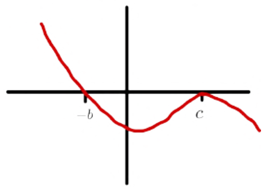
Roots : $x = -b, c$ (repeated so touches at $x = c$)

y-intercept : $ab(-c)^2 = abc^2$

$a > 0$, Shape: Uphill



$a < 0$, Shape: Downhill

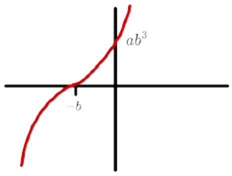


$y = a(x+b)^3$

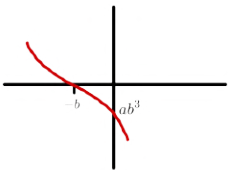
Roots : $x = -b$ (repeated three times so point of inflection at $x = b$)

y-intercept : $ab^3 = ab^3$

$a > 0$, Shape: Uphill



$a < 0$, Shape: Downhill



Point of inflection is neither a minimum or maximum but the gradient is zero.

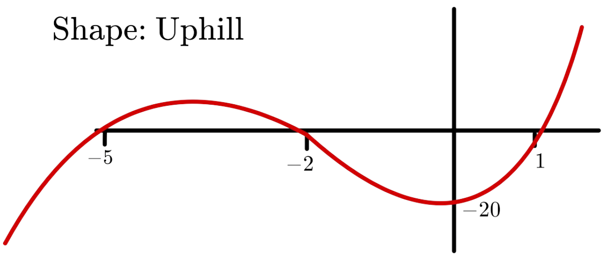
Example 1

Draw the graph $y = 2(x-1)(x+2)(x+5)$

Roots : $x = 1, -2, -5$

$y - intercept : (2)(-1)(2)(5) = -20$

Shape: Uphill



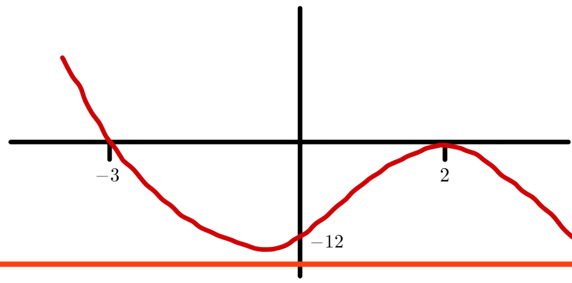
Example 2

Draw the graph $y = -(x+3)(x-2)^2$

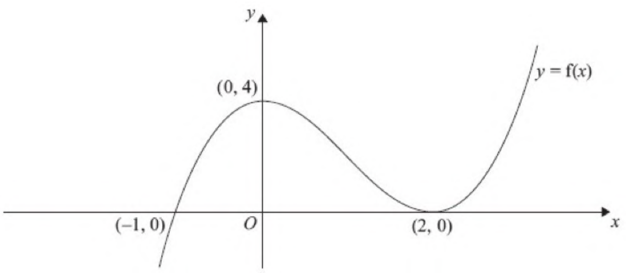
Roots : $x = -3, 2$ (repeated)

$y - intercept : (-1)(3)(-2)^2 = -12$

Shape: Downhill



Example 3



Find the equation of the graph

Roots : $x = -1, 2(\text{repeated})$

$\Rightarrow y = a(x+1)(x-2)^2$

$x = 0, y = 4$

$4 = a(1)(-2)^2 \Rightarrow a = 1$

$\Rightarrow y = (x+1)(x-2)^2$

Example 4

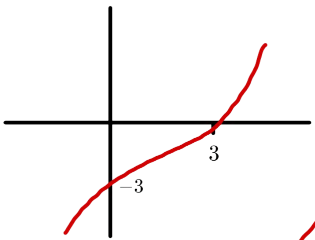
Draw the graph $y = (x-3)(x^2+x+1)$

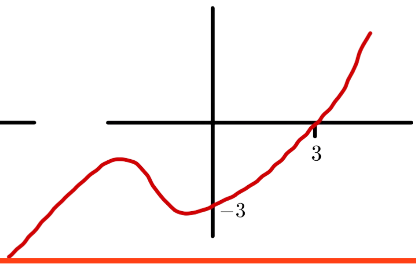
Roots : $x = 3$ only

$y - intercept : (-3)(1) = -3$

Shape: Uphill

Not enough information to draw the exact shape.
Two possibilities shown.

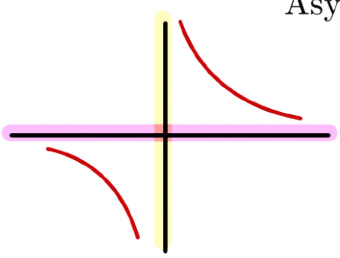




Reciprocal Graphs

$$y = \frac{k}{x} \quad , x \neq 0$$

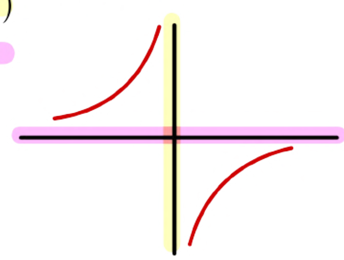
$k > 0$



Asymptotes : $x = 0\ (y - axis)$

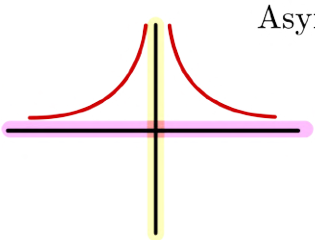
$y = 0\ (x - axis)$

$k < 0$



$$y = \frac{k}{x^2} \quad , x \neq 0$$

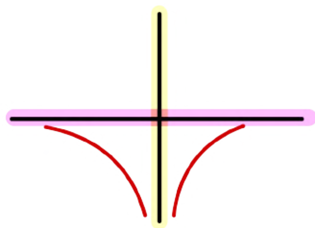
$k > 0$



Asymptotes : $x = 0\ (y - axis)$

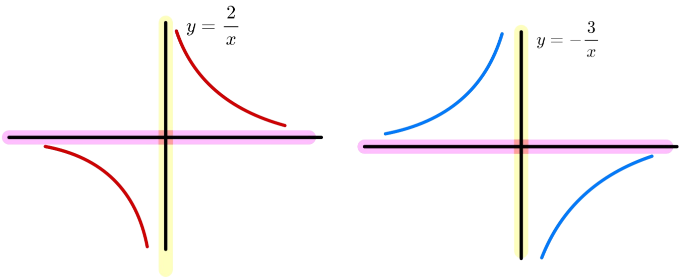
$y = 0\ (x - axis)$

$k < 0$



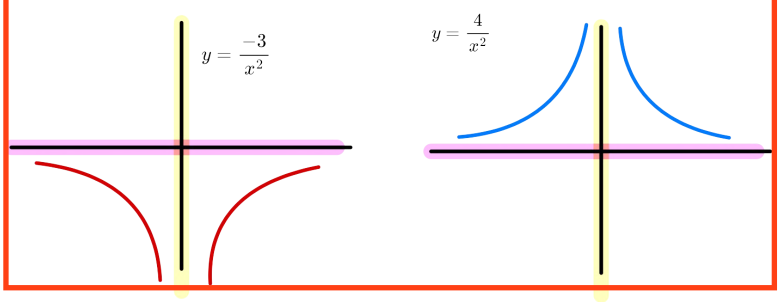
Example 1

Draw the following graphs : $\frac{2}{x}$ and $\frac{-3}{x}$



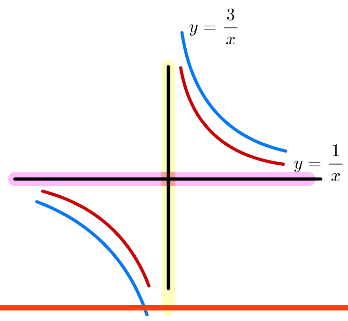
Example 2

Draw the following graphs : $\frac{-3}{x^2}$ and $\frac{4}{x^2}$



Example 3

Draw the following on the same graph : $\frac{1}{x}$ and $\frac{3}{x}$



Points of Intersection

If $y = f(x)$ and $g(x)$, x values of the points of intersection can be worked by solving $f(x) = g(x)$.

If $y = f(x) = g(x)$, if the graphs intersect n times, then the number of real solutions is n .

Example 1

Draw $y = x(x-1)$ and $y = x(x-2)(x+3)$ on the same graph and find the point of intersections.

$y = x(x-1)$

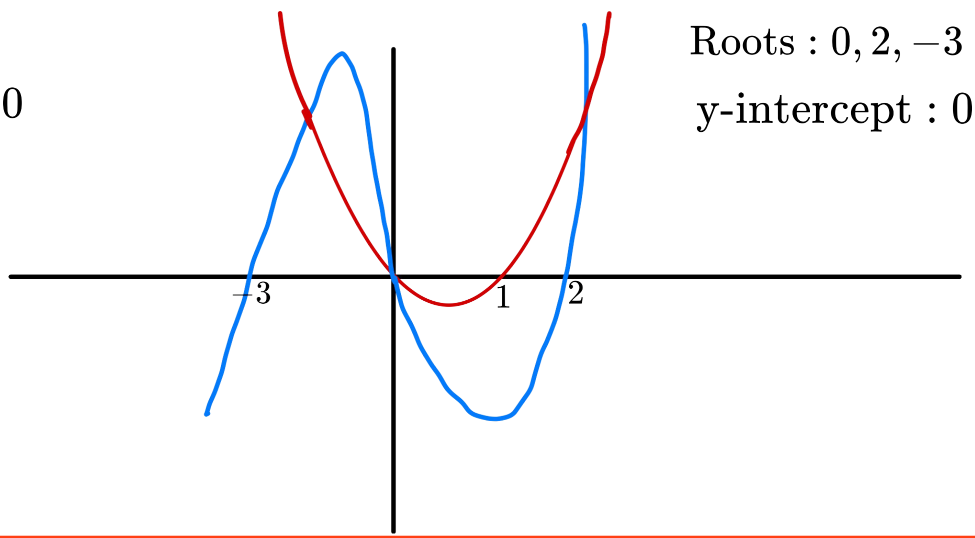
Roots : $0, 1$

y-intercept : 0

$y = x(x-2)(x+3)$

Roots : $0, 2, -3$

y-intercept : 0



$$x(x-1) = x(x-2)(x+3)$$

$$x(x-1) - x(x-2)(x+3) = 0$$

$$x((x-1) - (x-2)(x+3)) = 0$$

$$x((x-1) - (x^2 + 3x - 2x - 6)) = 0$$

$$x((x-1) - (x^2 + x - 6)) = 0$$

$$x(-x^2 + 5) = 0$$

$$x = 0$$

$$-x^2 + 5 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$$

$$x = 0, y = 0$$

$$x = -\sqrt{5}, y = 2\sqrt{5}$$

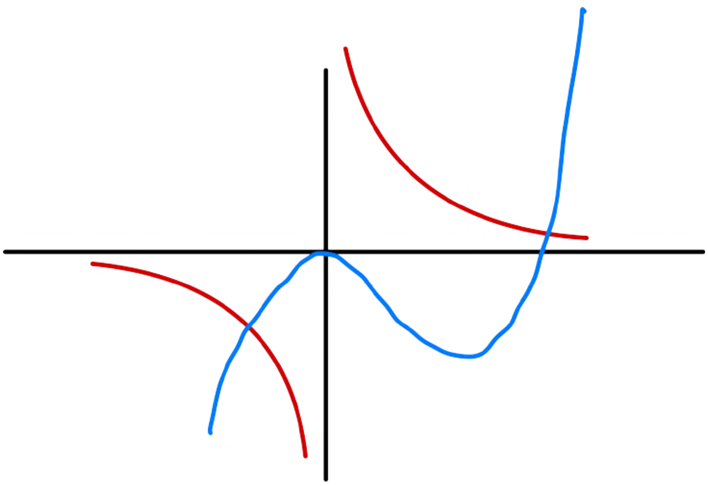
$$x = \sqrt{5}, y = 2\sqrt{5}$$

$(0,0)$, $(-\sqrt{5}, 2\sqrt{5})$ and $(\sqrt{5}, 2\sqrt{5})$ are points of intersection.

Example 2

Below is a sketch of $y = x^2(x-3)$ and $y = \dfrac{2}{x}$.

Find the number of real solutions to the equation $x^2(x-3) - \dfrac{2}{x} = 0$. Justify your answer



There are two insections points of the graphs therefore there are two solutions only.

Translation of Functions

If $y = f(x)$, then $y = f(x + a)$ is a translation of $f(x)$ by vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$

If $y = f(x)$, then $y = f(x) + a$ is a translation of $f(x)$ by vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$

If $y = f(x)$, then $y = f(x + a) + b$ is a translation of $f(x)$ by vector $\begin{pmatrix} -a \\ b \end{pmatrix}$

Example 1

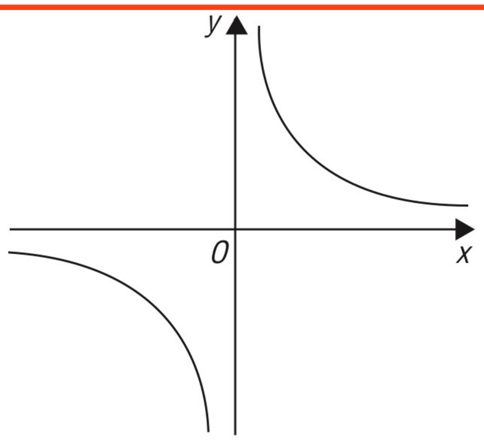


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \frac{3}{x}$, $x \neq 0$.

(a) On a separate diagram, sketch the curve with equation $y = \frac{3}{x+2}$, $x \neq -2$, showing the coordinates of any point at which the curve crosses a coordinate axis.

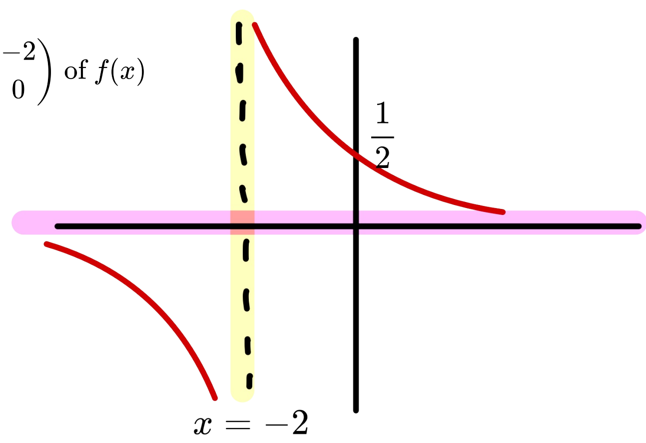
(b) Write down the equations of the asymptotes of the curve in part (a).

$f(x) = \frac{1}{x}$

$f(x+2) = \frac{1}{x+2}$ ⟵ Translation by vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ of $f(x)$

$x = 0 \, , y = \frac{1}{2}$

Asymptote is also translated



Asymptotes are $x = 2$ and $y = 0$

Example 2

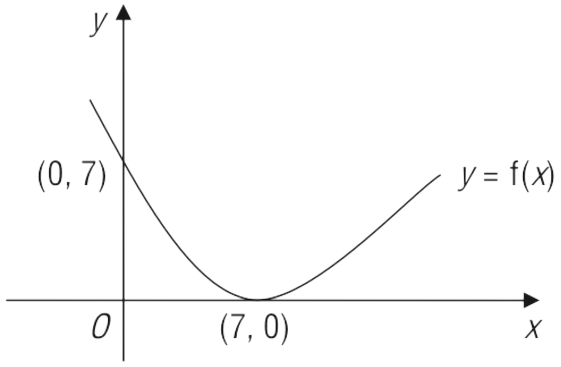


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$. The curve passes through the point (0, 7) and has a minimum point at (7, 0).

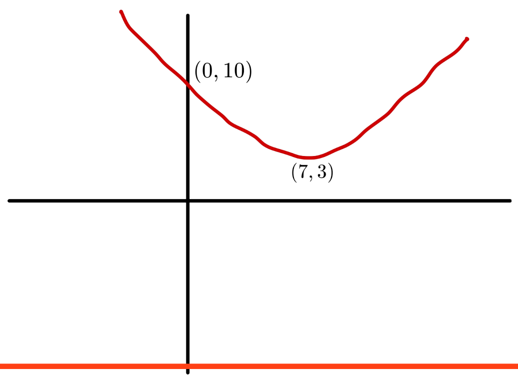
On separate diagrams, sketch the curve with equation

(a) $y = \mathrm{f}(x) + 3$,

Translation by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ of $f(x)$

$(0, 7) \to (0, 10)$

$(7, 0) \to (7, 3)$



Example 3

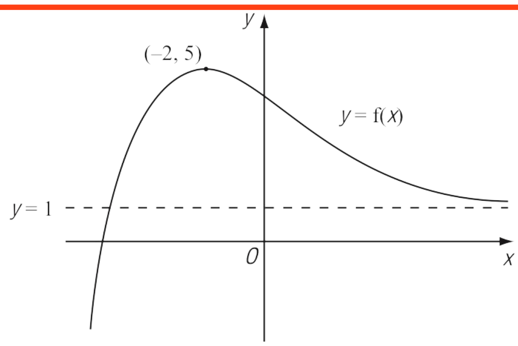


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$.

The curve has a maximum point $(-2, 5)$ and an asymptote $y = 1$, as shown in Figure 1.

On separate diagrams, sketch the curve with equation

(a) $y = \mathrm{f}(x) + 2$

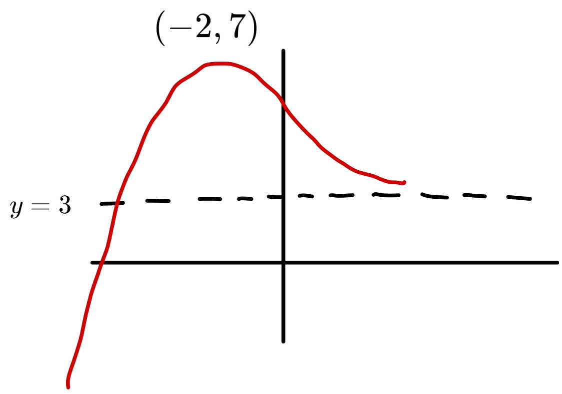
(b) $y = \mathrm{f}(x + 1)$

$a)$

Translation by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ of $f(x)$

$$(-2, 5) \rightarrow (-2, 7)$$

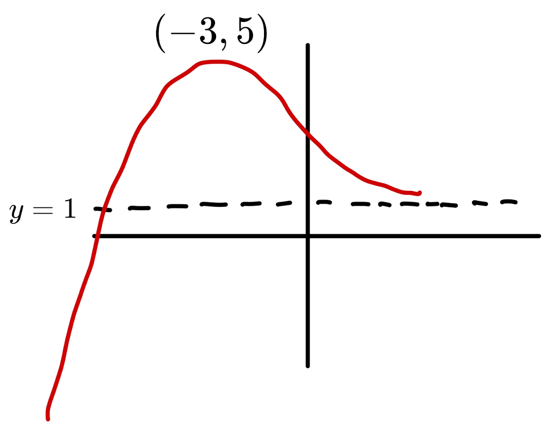
Asymptote : $y = 1 \rightarrow y = 3$



$b)$

Translation by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ of $f(x)$

$$(-2, 5) \rightarrow (-3, 5)$$



Stretching of Functions

If $y = f(x)$, then $y = f(ax)$ is a stretch horizontally of $f(x)$ by scale factor $\frac{1}{a}$

If $y = f(x)$, then $y = af(x)$ is a stretch vertically of $f(x)$ by scale factor a

Example 1

Figure 1

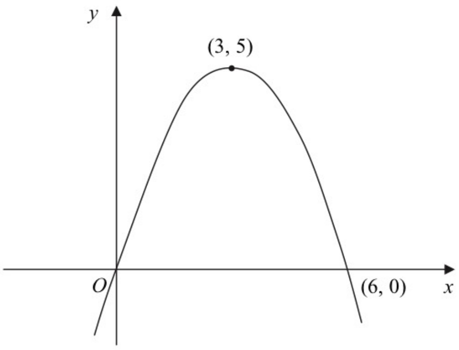


Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$. The curve passes through the origin O and through the point (6, 0). The maximum point on the curve is (3, 5).

On separate diagrams, sketch the curve with equation

(a) $y = 3\mathrm{f}(x)$,

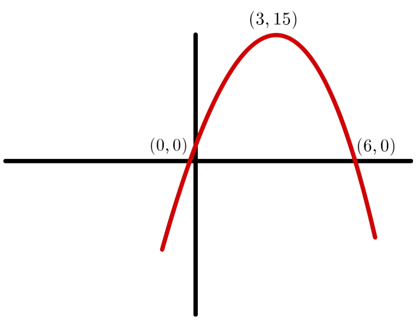
(b) $y = \mathrm{f}(2x)$.

$a)$ Vertical stretch by scale factor 3

$(0,0) \to (0,0)$

$(3,5) \to (3,15)$

$(6,0) \to (6,0)$

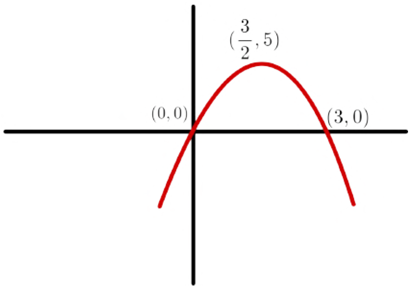


$b)$ Horizontal stretch by scale factor $\frac{1}{2}$

$(0,0) \to (0,0)$

$(3,5) \to (\frac{3}{2},5)$

$(6,0) \to (3,0)$



Reflections of Functions

If $y = f(x)$, then $y = f(-x)$ is a reflection of $f(x)$ in the y axis

If $y = f(x)$, then $y = -f(x)$ is a reflection of $f(x)$ in the x axis

Example 1

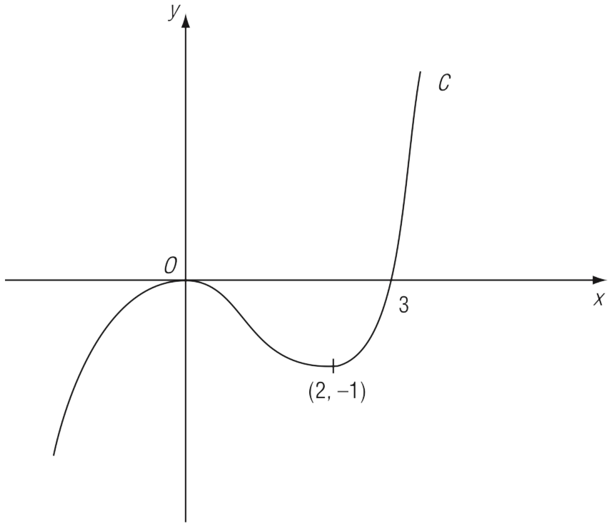


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = f(x)$. There is a maximum at (0, 0), a minimum at (2, –1) and C passes through (3, 0).

On separate diagrams sketch the curve with equation

(a) $y = f(-x)$.

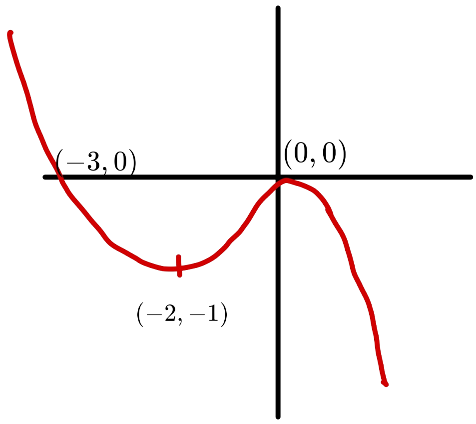
On each diagram show clearly the coordinates of the maximum point, the minimum point and any points of intersection with the x-axis.

$a)$ Reflection in the y axis

$(0,0) \rightarrow (0,0)$

$(2,-1) \rightarrow (-2,-1)$

$(3,0) \rightarrow (-3,0)$



Example 2

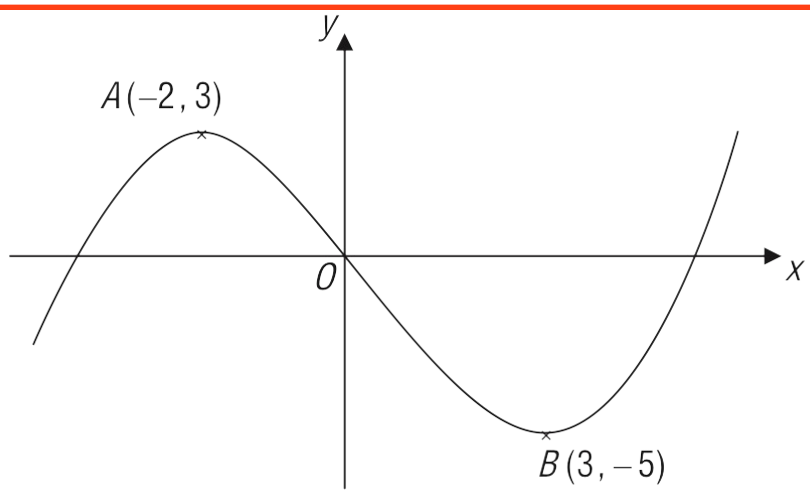


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$. The curve has a maximum point A at (–2, 3) and a minimum point B at (3, –5).

On separate diagrams sketch the curve with equation

(a) $y = -\mathrm{f}(x)$

On the diagram show clearly the coordinates of the maximum and minimum points.

$a)$ Reflection in the x axis

$$(0,0) \rightarrow (0,0)$$

$$(-2,3) \rightarrow (-2,-3)$$

$$(3,-5) \rightarrow (3,5)$$